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Research Article

Exploring Toeplitz, Circulant, and Hankel Matrices: Algebraic Structures and Field Properties with a Focus on Circulant Matrices

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ABSTRACT

This abstract provides an overview of the discussed topics related to Toeplitz matrices, Circulant matrices, Hankel matrices, and their connections to algebraic structures. Additionally, the abstract introduces the exploration of Group, Ring, and Field properties with a focus on the Field Properties of Circulant Matrices. Furthermore, the concept of Eigenvalues in Circulant Matrices is introduced, shedding light on the intrinsic characteristics of these matrices. The abstract concludes by highlighting the visualization aspect through the Graph of column vectors of Circulant Matrices, emphasizing the significance of graphical representations in understanding the matrix properties. Overall, the abstract encapsulates a comprehensive exploration of mathematical concepts and their applications, showcasing the interconnectedness of linear algebra and algebraic structures.

Keywords: Toeplitz matrix, Circulant matrix, Hankel matrix.

1) Introduction:

1.1) Toeplitz matrix

Toeplitz matrix or diagonal-constant matrix, named after Otto Toeplitz, is a matrix in which each descending diagonal from left to right is constant.
A $m \times n$ matrix is said to be a Toeplitz matrix whose descending diagonal entries are constant.

Example of Toeplitz matrix

$$A_1 = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 3 & 2 \\ 5 & 1 & 3 \end{bmatrix}_{3\times 3} \quad A_2 = \begin{bmatrix} 3 & 1 & 5 & 7 \\ 5 & 3 & 1 & 5 \\ 4 & 5 & 3 & 1 \\ 1 & 4 & 5 & 3 \end{bmatrix}_{4\times 4} \quad A_3 = \begin{bmatrix} 5 & 1 & 4 & 3 \\ 7 & 5 & 1 & 4 \\ 9 & 7 & 5 & 1 \end{bmatrix}_{3\times 4} \quad A_4 = \begin{bmatrix} 5 & 7 & 9 & 1 & 2 \\ 8 & 5 & 7 & 9 & 1 \\ 6 & 8 & 5 & 7 & 9 \\ 1 & 6 & 8 & 5 & 7 \end{bmatrix}_{4\times 5}$$

General Form of Toeplitz matrix

A Toeplitz matrix is an $n \times n$ matrix $T_n = \{t_{k,j}\,;\ k,j = 0,1,2,\dots,n-1\}$; where $t_{k,j} = k - j$ i.e., a matrix of the form

$$\begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & \cdots & & t_{-(n-1)} & \\ t_1 & t_0 & t_{-1} & t_{-2} & \cdots & & \vdots & \\ t_2 & t_1 & t_0 & \cdots & \cdots & & \vdots & \\ \vdots & \vdots & & \vdots & \ddots & \cdots & \cdots & \vdots \\ \vdots & \vdots & & \vdots & \cdots & \ddots & \cdots & \vdots \\ t_{n-1} & & \cdots & \cdots & \cdots & \cdots & \cdots & t_0 \end{bmatrix}_{n\times n}$$

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Application of Toeplitz matrix

Highly used in Signal Processing and Time Series analysis.
Toeplitz Matrix is, in general m × n matrix and may be a square matrix also.

Salient Features of Toeplitz matrix

m × n Toeplitz matrix need only m + n – 1 different entries.
n entries in first row and m entries in first column.
It is written as

$$[5 \quad 7 \quad 9 \quad 1 \quad 2 \quad / \quad 5 \quad 8 \quad 6 \quad 1]$$

First Row First Column

Note : First entry being common '5' . It has 4+5−1=8 different entries.

						Row Sum
$T_1 =$	5	$\boxed{7}$	9	1	2	24
	8	5	7	9	1	30
	6	8	5	7	9	35
	$\boxed{4}$	6	8	5	7	30
Column Sum	23	26	29	22	19	

Sum of elements of R_1 = 24 , Sum of elements of R_2 = 30 and $\sum R_2 - \sum R_1 = 30 - 24 = 6$
We have $|a_{15} - a_{21}| = |2 - 8| = |-6| = 6$
Similarly,
Sum of elements of C_1 = 23 , Sum of elements of C_2 = 26 and $\sum C_2 - \sum C_1 = 26 - 23 = 3$
We have $|a_{51} - a_{12}| = |4 - 7| = |-3| = 3$

Note that product of two Toeplitz matrices is not a Toeplitz matrix.

$$B = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 2 & 3 \\ 5 & 4 & 2 \end{bmatrix}_{3\times 3} \qquad C = \begin{bmatrix} 3 & 7 & 2 \\ 4 & 3 & 7 \\ 5 & 4 & 3 \end{bmatrix}_{3\times 3} \qquad B \cdot C = \begin{bmatrix} 13 & 19 & 22 \\ 35 & 46 & 31 \\ 41 & 55 & 44 \end{bmatrix}_{3\times 3}$$

1.2) Hankel Matrix

A matrix in which non - leading diagonals remain constant.

$$H_1 = \begin{bmatrix} 5 & 4 & 7 & 2 \\ 4 & 7 & 2 & 5 \\ 7 & 2 & 5 & 4 \\ 2 & 5 & 4 & 7 \end{bmatrix}_{4\times 4} \qquad H_2 = \begin{bmatrix} 9 & 8 & 7 & 1 \\ 8 & 7 & 1 & 9 \\ 7 & 1 & 9 & 8 \\ 1 & 9 & 8 & 7 \end{bmatrix}_{4\times 4}$$

	3 × 5					Row Sum
H_3 =	9	8	1	7	5	30
	8	1	7	5	4	25
	1	7	5	4	9	26
Column Sum	18	16	13	16	18	

Sum of each column elements equidistant from the first and the last remain same.

1.3) Circulant Matrix

A circulant matrix is defined as a special kind of Toeplitz matrix where each row vector is rotated one element to the right relative to the preceding row vector.

General form of circulant matrix

$$C = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & \cdots & t_{-(n-1)} \\ t_{-(n-1)} & t_0 & t_{-1} & t_{-2} & \cdots & \vdots \\ t_{-(n-2)} & t_{-(n-1)} & t_0 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & \cdots & \vdots \\ t_{-1} & t_{-2} & \cdots & \cdots & \cdots & t_0 \end{bmatrix}_{n\times n}$$

A Circulant matrix of order n × n has only 'n' different members.

$$P_2 = \begin{bmatrix} 5 & 4 & 8 \\ 8 & 5 & 4 \\ 4 & 8 & 5 \end{bmatrix}_{3\times 3}$$

It is a Toeplitz and Circulant matrix.
Every Circulant Matrix is Toeplitz but converse is not true
A Toeplitz matrix of order m×n is square matrix if $m = n$

1.4) Fundamental rule of Circulant Matrix

Multiplication by Scalar

If A is any circulant matrix and k∈r then multiplication of A by k denoted as kA
kA is also a circulant matrix

Note:

- If k = 0 then kA is a Null matrix denoted as 0 therefore Null matrix is a circulant matrix
- If k = −1 then kA = (−1) A is denoted as −A is known as additive inverse of A .

They must be of same order and $a_{ij} = b_{ij}$; ∀ i,j

Addition of two circulant matrices

If A and B are two circulant matrices then their sum denoted as A + B = C (say) is also a circulant matrix such that $a_{ij} + b_{ij} = c_{ij}$; ∀ i,j

2) Circulant, Toeplitz and Hankel Matrix

				Row Sum
C =	2	5	4	11
	4	2	5	11
	5	4	2	11
Column Sum	11	11	11	

det(C) = 77

2.1) Exmple of Hankel but not Circulant Matrix

				Row Sum
H =	2	5	4	11
	5	4	2	11
	4	2	5	11
Column Sum	11	11	11	

det(H) = −77

Here $|C|=77=k(\alpha)$ and $|H|=-77=k(\beta)$
Determinants differ by sign only and they are multiple of libra value.
Now, we establish algebraic structure like Group, Ring, Field etc..

3) Structure for Group with '+' on C

Let C be a Nonempty set of Circulant matrices and we define '+' on C

3.1) Properties of Circulant Matrix

Let C be a set of Circulant matrices with m × m elements and $c_1, c_2, c_3 \in C$ then
Associativity : $(c_1 + c_2) + c_3 = c_1 + (c_2 + c_3)$
Identity : C + 0 = 0 + C = C
Inverse : If c_1 is a circulant matrix then $-c_1 = (-1)c_1$ is also a circulant matrix.

$$\therefore c_1 + (-c_1) = (-c_1) + c_1 = 0$$

∴ $-c_1$ is an inverse of c_1
∴ (C,+) is a group

3.2) Commutative Property of Circulant Matrix

Let C be a set of Circulant matrices with m × m elements and $c_1, c_2, c_3 \in C$ then $c_1 + c_2 = c_2 + c_1$
∴ (C,+) is an abelian group under matrix addition

3.3) Product Property of Circulant Matrix

Let C be a set of Circulant matrices with m × m elements and $c_1, c_2, c_3 \in$ abelian group then their product denoted as c_1c_2 is also a circulant matrix.
(c_1 and c_2 are conformable for matrix multiplication under standard convention)

4) Ring of Circulant Matrix under (+,·)

Let C be a set of Circulant matrices with m×m elements and $c_1, c_2, c_3 \in C$ then

$$c_1 + c_2 \in C$$

Associativity : $(c_1 + c_2)+c_3=c_1+(c_2+c_3)$
Identity : $c_1 + 0 = 0 + c_1 = c_1$
Inverse : If c_1 is a circulant matrix then $-c_1 = (-1)c_1$ is also a circulant matrix.
∴ $c_1+(-c_1)=(-c_1)+c_1=\bar{o}$
∴ '$-c_1$' is an inverse of c_1
$c_1 \cdot c_2 = c_2 \cdot c_1$

$$c_1 \cdot (c_2 \cdot c_3) = (c_1 \cdot c_2) \cdot c_3$$
$$c_1 \cdot (c_2 + c_3) = (c_1 + c_2) \cdot (c_1 + c_3)\ c_1$$
$$+ (c_2 \cdot c_3) = (c_1 \cdot c_2) + (c_1 \cdot c_3)$$

∴ (C,+, ·) is a ring.

4.1) Commutative Ring of Circulant Matrix

Let C be a set of Circulant matrices with m×m elements and $c_1, c_2 \in C$ then $c_1 \cdot c_2 = c_2 \cdot c_1 \in C$ under standard matrix multiplication convention.

∴ (C,+, ·) is Commutative ring with identity I.
Identity matrix is also a circulant matrix of square order.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$$

Let $c_1 \in C$ then we define singularity if $\det(c_1) \neq 0$ then it is non - singular circulant matrix if $\det(c_1) = 0$ then there is a special class (zero class) circulant matrix
we have non zero divisor of circulant matrix

$$A = \begin{bmatrix} -2 & 3 & -1 \\ -1 & -2 & 3 \\ 3 & -1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \qquad A \cdot B = \begin{bmatrix} -7 & -7 & 14 \\ 14 & -7 & -7 \\ -7 & 14 & -7 \end{bmatrix}$$

4.2) Inverse of circulant matrix

Let $c \in (C, +, \cdot)$ and if $\det(c) \neq 0$ then its inverse exists and its denoted by c^{-1}

$$c^{-1} = \frac{1}{\det(c)} \operatorname{adj}(c)$$

Here inverse of c is also a circulant matrix $c = \begin{bmatrix} 3 & 7 & 2 \\ 2 & 3 & 7 \\ 7 & 2 & 3 \end{bmatrix}$

$$c^{-1} = \frac{1}{100} \begin{bmatrix} 2 & \frac{67}{10} & \frac{171}{10} \\ \frac{171}{10} & 2 & \frac{67}{10} \\ \frac{67}{10} & \frac{171}{10} & 2 \end{bmatrix}$$ determinant of C is

∴ (C,+, ·) is a field on R .
If we take set of complex number then also we have all the same properties satisfied over the field of complex number.

5) Eigen Value of Circulant Matrix (order - 4)

$$A = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \end{bmatrix}_{4\times4} \qquad A = [|4|, |3|, |2|, |1|] \qquad L(A) = 10 = \lambda_0$$

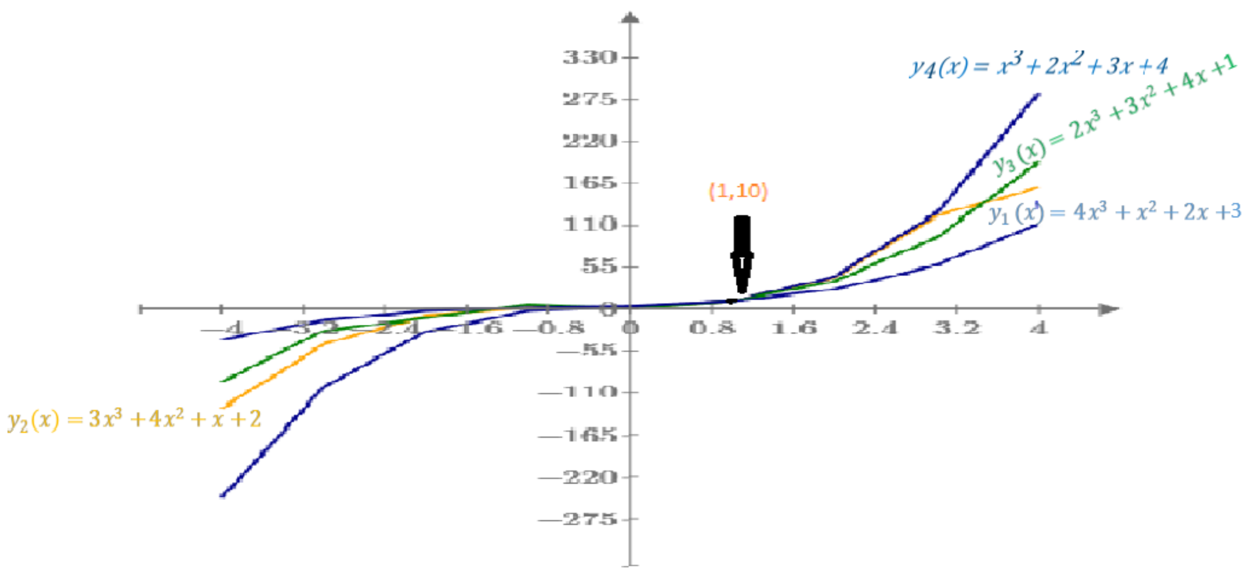
$$w = e^{i\,2\Pi(j/n)} \quad ; j = 1,2,3,4 \quad \text{and } n = 4$$

$$w = e^{i\,2\Pi(1/4)} = e^{i\,(\Pi/2)} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$
$$w^2 = e^{i\,2\Pi(2/4)} = e^{i\,(2\Pi/2)} = \cos\pi + i\sin\pi = -1$$
$$w^3 = e^{i\,2\Pi(3/4)} = e^{i\,(3\Pi/2)} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$$
$$w^4 = e^{i\,2\Pi(4/4)} = e^{i\,(2\Pi)} = \cos2\pi + i\sin2\pi = 1$$
$$w^6 = e^{i\,2\Pi(5/4)} = e^{i\,(5\Pi/2)} = \cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2} = i$$
$$w^9 = e^{i\,2\Pi(6/4)} = e^{i\,(3\Pi)} = \cos3\pi + i\sin3\pi = -1$$

one of the libra value is an eigen value of circulant matrix
$$\lambda_0 = 4 + 3 + 2 + 1 = L(A) = 10$$
$$\lambda_1 = 4 + 3w + 2w^2 + w^3 = 2 + 2i$$
$$\lambda_2 = 4 + 3w^2 + 2w^4 + w^6 = 3 + i$$
$$\lambda_3 = 4 + 3w^3 + 2w^6 + w^9 = 3 - i$$

$$y_1(x) = 4x^3 + x^2 + 2x + 3$$
$$y_2(x) = 3x^3 + 4x^2 + x + 2$$
$$y_3(x) = 2x^3 + 3x^2 + 4x + 1$$
$$y_4(x) = x^3 + 2x^2 + 3x + 4$$

5.1) Graph of Circulant Matrix Vector



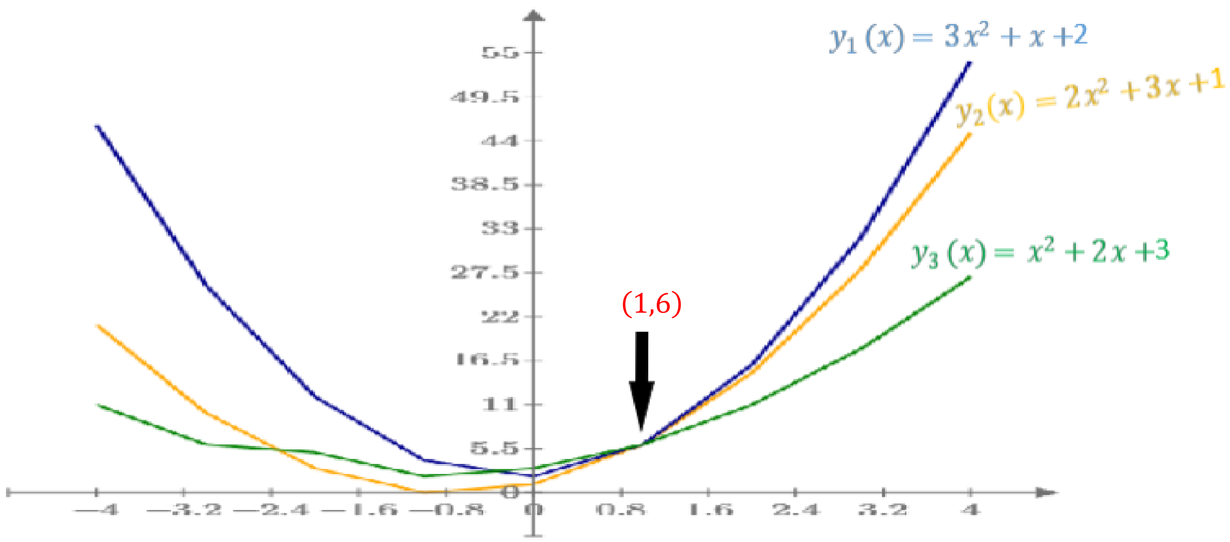
6) Eigen Value of Circulant Matrix (Order - 3)

$$P_2 = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}_{3\times3} \qquad P_2 = [|3|, |2|, |1|]$$

$w = e^{i\,2\Pi(j/n)}$; $j = 1,2,3$ and $n = 3$
$w = e^{i\,2\Pi(1/3)} = e^{i\,(2\Pi/3)} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \frac{1}{2}(-1+\sqrt{3}i)$
$w^2 = e^{i\,2\Pi(2/3)} = e^{i\,(4\Pi/3)} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \frac{1}{2}(-1-\sqrt{3}i)$
$w^3 = e^{i\,2\Pi(3/3)} = e^{i\,(2\Pi)} = \cos 2\pi + i\sin 2\pi = 1$
$w^4 = e^{i\,2\Pi(4/3)} = e^{i\,(8\Pi/3)} = \cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3} = \frac{1}{2}(-1-\sqrt{3}i)$
one of the libra value is an eigen value of circulant matrix
$\lambda_0 = 3 + 2 + 1 = L(P_2) = 6$
$\lambda_1 = 3 + 2w + w^2 = \frac{\sqrt{3}}{2}(\sqrt{3}+i)$
$\lambda_2 = 3 + 2w^2 + w^4 = \frac{\sqrt{3}}{2}(\sqrt{3}-i)$
$\lambda_3 = 4 + 3w^3 + 2w^6 + w^9 = 3 - i$
$y_1(x) = 3x^2 + x + 2$
$y_2(x) = 2x^2 + 3x + 1$

$\mathbf{y_3(x) = x^2 + 2x + 3}$

6.1) Graph of Circulant Matrix Vector



Conclusion:

We have explored the algebraic structures of Group, Ring, and Field, specifically highlighting the properties related to Circulant matrices within the context of Field theory.

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