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Research Article

A Comprehensive Review On Vertex Dominations In Graph Theory: Exploring Theory And Applications

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ARTICLE INFO	ABSTRACT

This comprehensive review delves into the intricate realm of VERTEX dominations in Graph Theory, providing an extensive exploration of both theory and practical applications. The article encompasses a thorough examination of various domination aspects in graphs, including Domination in Planar graphs, connected graph dominations, edge dominations in Paths, Cycles of related graphs, and associated properties. Additionally, the study extends to inverse dominations on graphs, shedding light on their significance in real-world scenarios. In graph theory, the idea of dominance states that a collection of vertices If every vertex in graph G is either in S or close to a vertex in S, then S dominates graph G. G's dominance number is based on the size of the least dominating set. In recent years, there has been interest in two alternative concepts: connected domination and absolute dominance. Every vertex in the graph must be next to every vertex in S for there to be a complete dominant set; nevertheless, a linked dominant set both dominates the graph and creates a connected subgraph. Numerous fields, including as radio programmes, computer communication networks, and school bus routing, may benefit from the use of these dominating concepts., social networks, and interconnection systems. The goal of the essay is to provide a comprehensive knowledge. of VERTEX dominations, establishing their theoretical foundations and illustrating their relevance in practical scenarios.

Keywords- Vertex Dominations, Graph Theory, Planar Graphs, Connected Graphs, Edge Dominations, Inverse Dominations, Domination Number, Total Domination, Connected Domination, Applications of Dominations.

I. INTRODUCTION

Graph theory, a vibrant field within modern mathematics, has witnessed remarkable growth in the past three decades. Its applications extend to several fields, such as classical algebraic problems, combinatorial issues, and discrete optimisation problems [1]. Furthermore, the influence of graph theory is evident in the realms of the social, biological, and physical disciplines, as well as linguistics. Among the various subfields of graph theory, the study of domination in graphs has emerged as a central and prolific area of research. The goal of the theory of dominance is to find dominating sets in graphs, investigate their characteristics, and comprehend the practical applications of these sets. When de Jaenisch studied the minimal number of queens needed to cover or control a nxn checkerboard. in 1862, the idea of dominance was born.[2]. Nonetheless, it wasn't until around 1960 that dominant sets in graph theory were thoroughly studied. The dominance number of a graph was first proposed by Berge in 1958 and was also known as the "coefficient of external stability." For the same idea, Ore later created the words "dominating set" and "domination number" in 1962. In 1977, Cockayne and Hedetniemi made a substantial addition to the field's knowledge of dominating sets when they published a thorough analysis of the findings about dominant sets in graphs at the moment. They invented the dominance number of a graph notation (G), which became widely used. The groundbreaking survey report by Hedetniemi and Cockayne sparked a surge in research activities related to domination in graphs. In the two decades following the survey, over 1200 research papers were published on this intriguing and multifaceted topic. This comprehensive review aims to delve into the intricacies of vertex dominations in graph theory, shedding light on various aspects such as domination insets, diverse types of domination, common minimal domination, theorems, outcomes and uses of dominance in graphs [3].

PRELIMINARY CONCEPTS

Graph: In terms of graph theory, a graph is a basic mathematical structure represented as an ordered triple, usually written as G=(V(G), E(G), IG). In this case, IG is an incident map, E(G) is a set disjoint from V(G), and V(G) is a non-empty set of vertices. Every element of E(G) is linked by the ensuing map to a previously order pair of items from V(G) that are either the same or different.[4]. This conceptualization lays the foundation for the study of relationships and connections within a network.

Vertices and Edges: The constituents of a graph include vertices, nodes, or points, denoted as V(G), and edges or lines, represented by E(G)[5] .The set V(G) comprises the vertices, while E(G) consists of the edges. For any edge e in E(G), if u and v are vertices such that IG(e)=uv, then e is deemed to join u and v. Furthermore, u and v are referred to as the ends of e, and the edge e is incident with these ends. Simultaneously, the vertices u and v are incident with the edge e. This nuanced terminology establishes the essential concepts of incidence and connection within the framework of a graph.

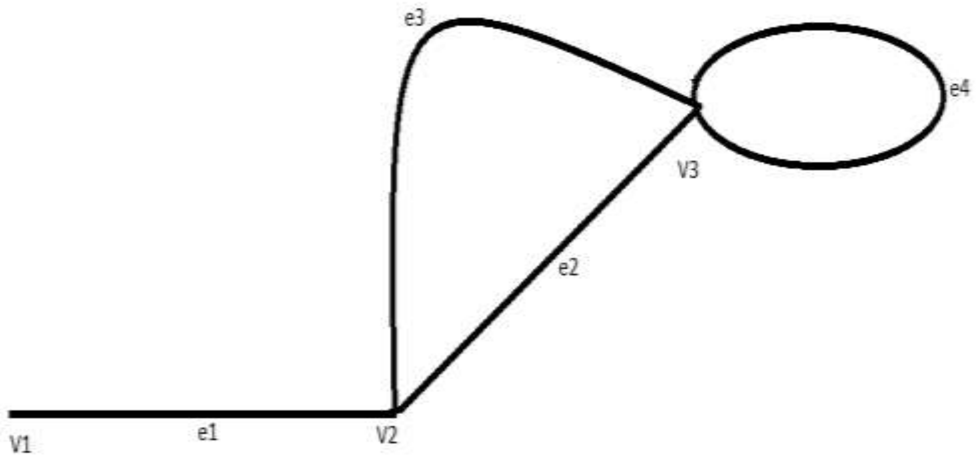


Figure No.1 Preliminary Concepts

V (G) = {v1, v2, v3}
E (G) = {e1, e2, e3, e4}
I_G(e1) = v1 v2
I_G (e2) =v2 v3 I_G(e3) =v3 v2 I_G(e4) =v3

Graph theory serves as a fundamental framework for modeling relationships and connections in various fields. Before delving into the topic of Vertex Dominations, it's essential to understand some preliminary concepts in graph theory.

Subgraph: The subgraph H H⊆G, the symbol for a graph G, is a key idea in graph theory. This means that V(G) is a subset of the edge set E(H) and the vertex collection V(H). subset of E(G)[6] Notably, if V(H) equals V(G), H transforms into a using G's subsection, intricately interwoven with the entirety of G by covering all its vertices. This fundamental relationship forms the basis for exploring vertex dominations, a multifaceted aspect in graph theory. Understanding such subgraphs and their interplay within a larger graph structure is essential for unraveling the theoretical underpinnings and practical applications of vertex dominations, promising insights into network analysis and optimization problems.

Parallel Edges and Loops: In In the context of graph theory, the existence of edge pairs with looping adds a layer of structural complexity. Parallel edges, exemplified by instances like e2 and e3, denote the occurrence of two or more edges sharing identical end vertices[7]. Meanwhile, loops, represented by edges such as e4, arise when an edge connects a vertex to itself. This duality of parallel edges and loops contributes significantly to the overall diversity and intricacy of graph structures. Understanding and analyzing these phenomena are essential for exploring the nuanced aspects of vertex dominations in graph theory and their applications in various real-world scenarios.

Link and Neighbourhood: In graph theory, the analysis of vertex dominations plays a pivotal role. One fundamental concept is the identification of edges with distinct end vertices, termed as links (e.g., e1, e2, excluding loops)[8]. The neighborhood of a vertex v, symbolized as N[V], comprises all vertices adjacent to v, constituting the open locality. Moreover, the closed neighbourhood, represented by N[V], includes both the open neighbourhood and and the vertex v itself. This distinction between open and closed neighborhoods is crucial for understanding connectivity patterns and influence propagation within a graph. As we delve into a comprehensive review of vertex dominations, exploring these foundational concepts provides a solid groundwork for the theoretical framework and practical applications in diverse domains.

Adjacency and Simple Graph: An essential idea in graph theory is the adjacency of edge and vertex. A graph with G has two vertices., represented by the letters u and v, are said to be nearby if an edge connects them.[9]. Likewise, If two different edges, denoted by e and f, have a shared end vertex, they are considered neighbouring. To enhance clarity and facilitate analysis, a graph is characterized as simple when devoid of loops or parallel edges. This simplicity in structure not only aids in theoretical exploration but also proves valuable in practical applications. As we delve into the comprehensive review on vertex dominations in graph theory, understanding these foundational concepts becomes crucial for unraveling the intricacies of graph structures and their diverse applications.

Finite and Infinite Graphs: A fundamental distinction arises based on the finiteness of a graph. A graph is deemed finite when The edge set (E(G)) and vertex set (V(G)) are each of limited size. cardinality; otherwise, it assumes the classification of an infinite graph. The order of a graph, represented by n(G), encapsulates the count of its vertices, while the size, denoted as m(G) or simply n, enumerates the edges within the graph[10]. These foundational concepts form the bedrock for the exploration of vertex dominations in graph theory. In order to fully understand the complexities of this topic, it is necessary to explore the theoretical underpinnings and real-world applications, thereby unraveling the nuanced interplay between vertices within the graph structure.

Degree of Vertices and Regular Graphs: The quantity of edges in a graph G that are incident to a vertex v is known as its degree in graph theory, and it is denoted as dG(v). This basic idea is essential to understanding the structural characteristics of graphs. A graph G's lowest and maximum degrees, represented by the symbols $\delta(G)$ and $\Delta(G)$, respectively, provide information on the graph's connectivity and intricacy.[11]. A graph is deemed K-regular when each vertex possesses a consistent degree K, and it attains the status of a regular graph if it is K-regular for a non-zero K. This notion of regularity serves as a cornerstone for understanding and analyzing various graph structures, laying the groundwork for exploring the rich landscape of vertex dominations in graph theory. Isolated Vertex and Leaf: An isolated vertex has a degree of zero, meaning it is not an endpoint of any edge. A leaf (or pendent) vertex has a degree of one, connected to only one other vertex. Understanding the concepts of isolated vertices and leaves is crucial. An isolated vertex, characterized by a degree of zero, signifies its lack of connection to any edge endpoint. On the other hand, a leaf or pendent vertex, with a degree of one, is linked to only a single neighboring vertex[12]. These fundamental notions lay the groundwork for more intricate discussions on vertex dominations, enriching our comprehension of graph structures and their diverse applications.

DOMINATING SET

The exploration of dominating sets holds paramount significance for unraveling the intricacies of graph structure and connectivity[13]. A dominating set comprises vertices strategically positioned to exert control over the entire graph. This concept is pivotal in comprehending the dynamics of graphs from various perspectives, elucidated through three fundamental definitions. The first definition encapsulates the notion of a a collection of vertices known as the dominant set where each vertex is one of two a member of The group, or close by. This second definition extends this by emphasizing the minimality of the dominating set. Lastly, the third definition introduces the idea of redundancy, emphasizing the uniqueness of dominance. This comprehensive review delves into the nuances of vertex dominations, offering a profound exploration of their theoretical underpinnings and diverse applications.

Definition-1:

Graph TheoryWith a graph G= (V, E), a dominant set $D \subseteq V$ holds a crucial role, defined by the condition that each vertex u in V-D, a neighbour $v \in D$ is present [14]. This essential idea highlights the significance of a dominating set in covering the entirety of the graph, leaving no vertex untouched by its influence. The exploration of vertex dominations extends beyond mere theoretical implications, offering practical applications in diverse fields. Whether in network design, social network analysis, or optimization problems, the profound implications of vertex dominations make it a subject of comprehensive review, delving into both its theoretical underpinnings and its wide-ranging applications.

Definition-2:

In graph theory, the neighbourhood of D, written as NG[D], contains the whole collection of vertices V(G). This is known as a dominant set, and it is expressed as D⊆V in a graph G=(V, E). A dominant set, to put it simply, guarantees that each vertex in V(G) is either directly contained in D or related to a vertex in D.[15]. This definition sheds light on the far-reaching impact of a dominating set within the graph, offering an alternative lens through which to understand its influence. Exploring the nuances of vertex domination not only contributes to theoretical advancements in graph theory but also finds practical applications in various fields, making it a subject of comprehensive study and analysis.

Definition-3:

In graph theory, a dominant set emphasises the idea of control within the graph. It is defined as a subset D of vertex in a graph G= (V, E) where Any vertex outside of D is next to a minimum single D component..[16]. This set effectively governs the entire graph, exerting influence over non-member vertices through adjacency relationships. Understanding these definitions serves as a foundational step in comprehending the critical role dominating sets play in graph theory. This comprehensive review delves into the multifaceted realm of vertex dominations, aiming to unravel their theoretical underpinnings and diverse practical applications across domains. By scrutinizing the intricacies of dominating sets, researchers and practitioners gain insights into the fundamental structures that govern graph dynamics. From network design to optimization problems, the study of vertex dominations proves invaluable in addressing real-world challenges and enhancing our understanding of graph-based systems[17]. This exploration forms the cornerstone for unlocking the potential of vertex dominations in both theoretical frameworks and applied scenarios.

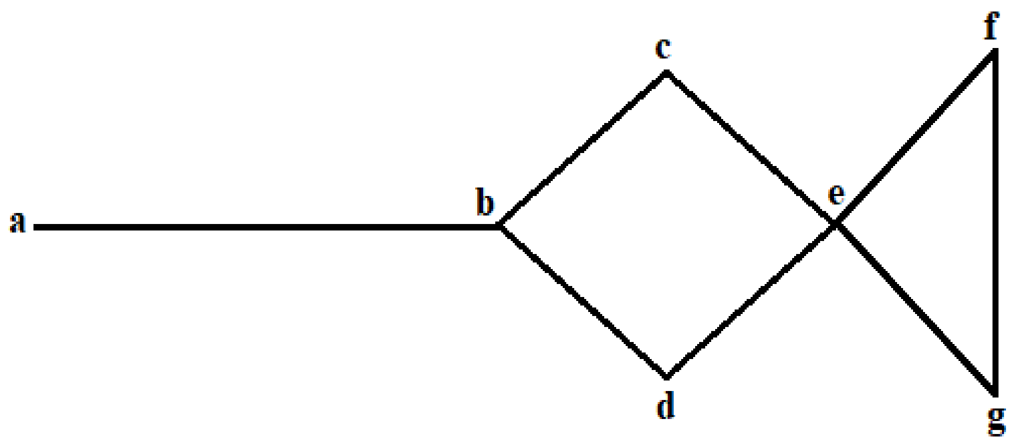


Figure No.2 Dominating Set

Graph theory, a branch of discrete mathematics, plays a crucial role in modeling and understanding various real-world systems[18]. One fundamental concept in graph theory is the notion of vertex domination, which has widespread applications in diverse fields such as network design, communication systems, and optimization problems. In this comprehensive review, we delve into the intricacies of vertex dominations, with a focus on dominating sets, minimal dominating sets, minimum dominating sets, domination numbers, and related theorems[19].

Dominating Set

Dominating sets play a crucial role in graph theory, in which A dominating set is the subset of edges in a graph that ensures each vertex is either a member of the set or is next to a minimum of a single member of the set. [20]. For instanceConsidering the graph G, each vertex is either b, g, or close to one of the sets {b, g}, making it a dominant group them. Another illustration is the dominating set {a, b, c, d, f} in graph G. This comprehensive review delves into the intricacies of vertex dominations in graph theory, exploring both theoretical foundations and practical applications, shedding light on the significance of these sets in diverse domains.

Minimal Dominating Set

An important idea is that of vertex dominations, or more precisely, minimum dominating sets. When any vertex is removed from a dominant set D, in which case the set is no longer regarded to be a dominating set, the set is said to be minimal. This means that the set D - v is no longer a dominant set for each vertex v in D. It is important to make sure that the dominant set is as compact as possible. Examples of minimum dominant sets in the context of a particular graph (Figure 2) are {b, e} and {a, c, d, f}. Gaining an understanding of and investigating such sets is essential to deciphering the complex dynamics of graph theory and finding useful applications across a range of fields.[21].

Minimum Dominating Set

A minimal group that dominates is defined as a dominating set that consists of the fewest possible vertices[22]. Examining Figure 2, the set {b, g} stands out as a minimally prevailing set, as it encompasses the smallest quantity of vertex compared to all other dominating sets. This concept is crucial in the study of graphs, where the objective is to identify the smallest portion of the vertex with efficiently control the entire graph. The significance of minimum dominating sets lies in their ability to optimize the use of vertices while maintaining dominance, contributing to efficient graph analysis and problem-solving strategies. Understanding and identifying such sets play a key role in various applications, ranging from network design to resource allocation.

DOMINATION NUMBER

The lowest number of vertices needed to establish a dominating set is known as the domination number (γ(G)) for a given graph G. For instance, in Figure 2, γ(G) = 2 is obtained since the smallest dominant set {b, e} has two entries. This parameter is a quantitative indicator that measures how well dominant sets cover the whole graph. In essence, it is the minimal vertex set size required to guarantee that each vertex in the graph is either adjacent to or a member of the dominant set. One key idea in graphs is the dominance number. theory, offering insights into the structural characteristics and resilience of a given graph[23].

Theorem 1.1:

This theorem establishes requirements for the minimality of a dominant set. If S is a minimum dominant set of the graph G, then S is such if and only if each vertex u in S satisfies one of those terms: One of two possibilities occurs: (i) u is an island in S, meaning that u's neighbourhood in S is empty; (ii) the vertex v exists in the complements of S (V - S) such that u's neighbour in S is {u}.

Proof: The proof involves considering the dominant set's minimalism and exploring the conditions under which it remains minimal. Dominating sets plays a vital part in understanding and analyzing graph structures. In the context of the comprehensive review on vertex dominations in graph theory, one fundamental concept to explore is the notion of a minimal dominating set. Let's delve into the details. If the set S - {u} is not the dominant set of G for each vertex u in S, then the dominant set S in graph G is deemed minimum. [24]. This condition implies that for each vertex u in S, there exists a vertex v in the complement of S (V - S) such that the neighborhood of v, denoted as N(v), does not intersect with the set S - {u}. To establish the minimality of S, two conditions must be satisfied. First, if v equals u, condition (i) is met. Second, if v does not equal u, the neighborhood of v in the union of S and {u} (N(v) intersect S ∪ {u}) is an empty set, fulfilling condition (ii). These conditions ensure the non-existence of a smaller dominating set in G. Conversely, if a dominating set S is not minimal, there exists a subset S' ⊆ S such that S' is also a dominating set[25]. In this case, a vertex u in S - S' can be identified. Removing u from S does not affect the domination property, indicating that S - {u} is still a dominating set. The presence of u as a non-isolated vertex in S - {u} ensures that conditions (i) and (ii) are satisfied for this specific u in S.

Theorem 1.2:

This theorem states that if a If S is a minimum dominant set of graph G and there are no isolation vertex in graph G, then the inverse of S (V - S) is also the least dominant set of G.

Proof: The proof involves assuming the contrary and demonstrating a contradiction, showcasing that the complement of S is indeed a dominating set. In the theory of graphs, a dominant set acts a crucial role in understanding the connectivity and control within a graph. In the context of the comprehensive review on vertex dominations, let's delve into the concept of a dominating set, shedding light on its implications and applications[26]. Consider a vertex u in a graph, and let S be a dominating set. The claim is that the neighborhood of u, denoted as N(u), is not a subset of S. To understand this, let's assume the contrary, where N(u) is a subset of S. In such a scenario, u must be adjacent to a vertex in S, and any vertex outside of S must be adjacent only to vertices in S - u. This condition implies that the set S - u is itself a dominant set, which goes against S's minimality. To put it another way, for each vertex u in S, S cannot encompass the neighbourhood of u (N(u)) completely. In the vertex set (V - S), the intersection of N(u) and S's complement is also non-empty. This suggests that the vertex set's counterpart of S, denoted as V - S, serves as a dominating set[27].

Corollary:

If a command n a graph G (number of vertices) contains no disconnected vertices, the domination number γ(G) is less than or equal to 2.

Proof: The proof utilizes the concept of minimum dominating sets and their complement to determine a maximum limit for the dominance value in graphs without isolated vertices[28]. In Within graph theory, the field concept of dominating sets plays a pivotal role, particularly in the exploration of vertex domination. Consider a graph G and let S be a minimum dominating set, denoted as γ(G). According to the theorem, the complement of S in the vertex set V(G), represented as V(G)-S, also forms a dominating set for G.

Consequently, the cardinality of the dominating set, |S|, is less than or equal to the cardinality of its complement, |V(G)-S|, leading to the inequality n γ(G) ≤ |V(G)-S| = n-γ(G). Simplifying this expression, we arrive at 2γ(G) ≤ n, which implies hence γ(G), the dominant set, has a size for bounded by half the number of vertices in G. This intriguing relationship, γ(G) ≤ 2, underscores the significance of dominating sets in understanding the structural characteristics of graphs. As we delve deeper into the theoretical aspects, it becomes evident that these insights have practical implications and applications in various fields. The study of vertex domination transcends mere mathematical abstraction, finding relevance in real-world scenarios where optimization and efficient resource allocation are paramount. Thus, the exploration of dominating sets not only enriches graph theory but also opens avenues for practical applications in diverse domains. The study of vertex dominations in graph theory is rich with theoretical implications and practical applications. Dominating sets and their variants provide valuable insights into the structural properties of graphs, offering solutions to optimization problems and aiding in the design of efficient network architectures. The theorems and corollaries presented in this review contribute to a deeper understanding of vertex dominations and pave the way for further exploration in this fascinating field[29].

VARIETIES OF DOMINATIONS: COMMON MINIMAL DOMINATION

Common Minimal Domination

In the realm of graph theory, a dominating set for a graph G= (V, E) is a subset D of vertex from V such that every vertex in V is either in D or close to a vertices in D. A set is considered small if removing one vertex from a dominating set makes it not dominant. The domination number, or γ(G), is the lowest cardinality of a dominating set in G. In contrast, the upper dominion number, Γ(G), is the largest cardinality over all minimal dominant sets in G.

Neighbourhood Graph (N(G))

The neighborhood graph N(G) is a construct derived from a graph G, sharing the same vertex set. In N(G), vertices are deemed adjacent only if they have been linked by a neighbor in the original graph G. This concept is pivotal for comprehending the intricate relationships and connections among vertices within the graph. By focusing on shared neighbors, N(G) provides a refined perspective on local structures, facilitating the analysis of proximity and influence among graph elements[30]. Understanding the neighborhood graph enhances graph theory applications, aiding in tasks such as pattern recognition, social network analysis, and the exploration of interconnected systems where vertices' interactions play a crucial role in deciphering underlying patterns and behaviors.

Common Minimal Dominating Graph (CD(G))

The typical minimum dominant graph A fascinating expansion of the dominant set idea is provided by CD(G). The CD vertex set (G) in this build is mirrored by that of G. Interestingly, two vertices in CD(G) are only considered neighbouring if there is a minimum dominant set in G that includes both of them. This construction sheds light on the intricate relationship between minimal dominating sets and the structural nuances of the original graph. Figure 3 below vividly illustrates a graph G alongside its corresponding common minimal dominating graph CD(G), offering a visual representation of this insightful interplay and further emphasizing the significance of CD(G) in exploring the underlying properties of dominating sets in graphs[31].

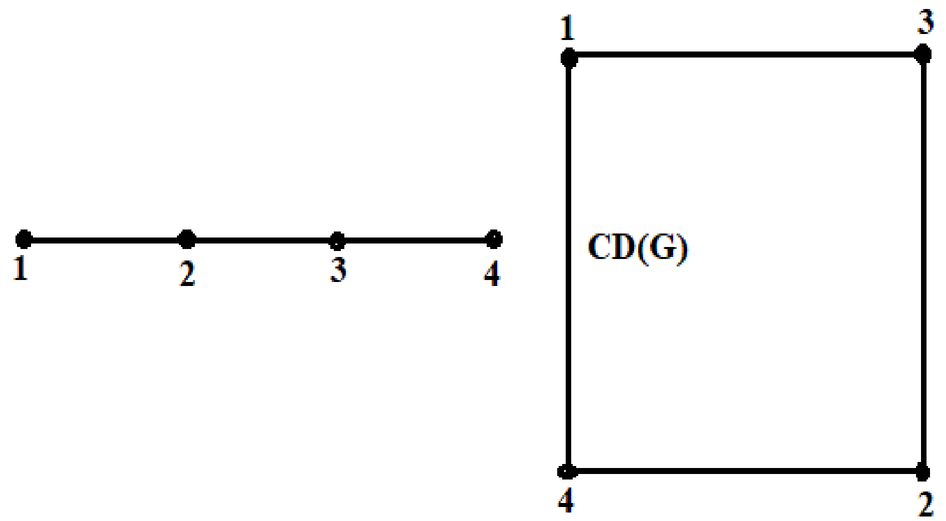


Figure No.3 Varieties of Dominations

Mathematicians who study graph theory look at the connections between items that are shown as vertices and edges.. One significant aspect of graph theory is vertex domination, which has numerous applications in various domains. In this review, we delve into the theory and applications of vertex domination, focusing on varieties of domination and providing a detailed analysis of theorems related to this concept.

Varieties of Domination

Theorem:1 In graph theory, for any graph G, the complement graph G' has a property known as the Complement Domination (CD)[32] Specifically, G' CD(G), indicating that the complement graph dominates the original graph. Notably, G' equals CD(G) if and only if every minimal dominating set of G is independent. This implies that in G', the vertices selected for domination do not share any common edges. This property plays a crucial role in understanding the relationships between dominating sets and independence in graphs. The concept of Complement Domination and its equivalence to independent minimal dominating sets contribute to the comprehensive study of graph structures and their properties.

Proof: The theory that is put out reveals an important connection between graph dominance and its complement. An edge (uv) in a graph (G') indicates that the vertices u and v may be enlarged to form a maximal isolated set S within group G if it is a component of the complement of the dominating graph. This relationship emphasises how crucial it is to comprehend how independence and dominance interact in order to fully comprehend the complex structure of domination graphs. Through the investigation of these connections, scholars acquire significant understanding of the underlying dynamics of graph theory, augmenting our understanding of the basic laws regulating graph structures and their complement counterparts.

Theorem:2 For any graph with p vertices ($p \geq 2$), CD(G) is connected if and only if $\Delta(G) < p-1$.

Proof: The theorem under consideration provides a criterion for determining the connectedness of the complement of the domination graph. The pivotal factor in this determination is the maximum degree of the original graph, denoted as $\Delta(G)$. The connectedness of the complement CD(G) is contingent upon the relationship between $\Delta(G)$ and the parameter p-1. Specifically, if the maximum degree $\Delta(G)$ is less than p-1, then the complement CD(G) is established as a connected graph. This result underscores the significant role that the maximum degree of the original graph plays in influencing the structural properties of the complement of the domination graph.

Theorem:3 For any graph G, $\gamma(CD(G)) \leq \omega(G)$.

Proof: The domination number, denoted as $\gamma(G)$, characterizes the minimum number of vertices in a graph G that can The presented theorem addresses the upper limit of the domination number concerning the complement of the domination graph (CD(G)). dominate or control the entire graph. This mathematical concept is intricately linked to the independence number $\beta_0(G)$, which represents the size of the largest independent set of vertices in G—meaning no two vertices in the set are adjacent. Additionally, the clique number $\omega(G)$, indicating the size of the largest clique in G, also plays a role in understanding the relationships within this mathematical framework. The theorem contributes to the broader study of graph theory by elucidating constraints on the domination number in relation to these graph parameters[33].

Theorem:4 For any graph G, $\gamma(CD(G)) \leq p-\Gamma(G)+1$.

Proof: The domination number of the complement of the domination graph is intricately connected to the domination number and order of the original graph[34]. A theorem that creates a discrepancy among three crucial parameters—the order of the graph p, the dominant number $\Gamma(G)$ of the original graph, and the domination number $\gamma(CD(G))$ of the complement graph—capsulates this connection. This theorem clarifies how these elements interact, providing information on the structural characteristics of graphs. In graph theory, knowing how these characteristics relate to one another is essential for gaining a deeper understanding of graph topologies and its dominant sets.

Theorem:5 For any graph G, $\gamma(CD(G)) \leq 1+\delta(G)$.

Proof: The theorem under consideration establishes a crucial link between the domination number of CD(G) and the minimum degree of the original graph, denoted as $\delta(G)$[35]. This connection is an extension of Theorem 1, which already establishes a relationship between the domination numbers of a graph G and its complement. In essence, the theorem provides valuable insights into how the domination number of the Cartesian product of a graph with its complement is influenced by the minimum degree of the original graph. This type of mathematical relationship is fundamental in graph theory, contributing to a deeper understanding of structural properties and interconnections within graphs.

Theorem:6 If G is an odd graph with Γ(G)=diam(G)=2, then CD(G) is eulerian.

Proof: This theorem delves into the Eulerian characteristics of the complement of the domination graph within a distinct category of graphs, namely odd graphs with a diameter of 2. In graph theory, domination graphs represent relationships between vertices, and their complements offer valuable insights into the structural intricacies of such graphs. Odd graphs, characterized by vertices with odd degrees, add a unique dimension to this exploration. The focus on graphs with a diameter of 2 suggests a specific connectivity pattern, emphasizing the importance of proximity in the studied graphs. Understanding the Eulerian aspects of these complements sheds light on the underlying patterns and connectivity properties, contributing to the broader field of graph theory and its applications[36].

Theorem:7 Let G be a graph of order at least three satisfying specific conditions; then CD(G) is Hamiltonian.

Proof: The theorem in question lays out the criteria for the complement of the domination graph to be Hamiltonian[37]. The dominating graph, denoted as G, exhibits a pivotal connection between the degree and neighborhood structure of its vertices, influencing the Hamiltonian characteristics of its complement, CD(G). The degree of vertices and the interplay of their neighborhood structures are integral factors in unraveling the Hamiltonian properties of CD(G). This theorem provides valuable insights into the relationship between graph theory and Hamiltonian cycles, shedding light on the intricate dynamics of domination graphs and their complements. The findings contribute significantly to the broader understanding of graph theory and its applications in mathematical concepts and problem-solving.

TOTAL DOMINATION

Definition:

Total domination in graph theory is intricately linked to dominating sets. A set S ⊆ V (the vertex set of graph G) qualifies as a total dominating set when each vertex in V is adjacent to at least one vertex in S. Another perspective defines a dominating set D as a total dominating set if the induced subgraph G[D] contains no isolated vertices. This concept plays a crucial role in analyzing the efficiency and connectivity of networks represented by graphs. Total dominating sets contribute to comprehending the interdependence of vertices in ensuring coverage and cohesion within the broader context of graph structures, aiding in diverse applications such as network design, communication protocols, and optimization algorithms.

Total Domination Number (γt):

The total domination number, denoted as γt(G), of a graph G signifies the size of the smallest total dominating set within G. This set, commonly known as a γt-set, plays a crucial role in graph theory[38]. Notably, the total domination number (γt) is inherently larger than or equal to the domination number (γ) for any graph G. A total dominating set in a graph is characterized by its ability to cover every vertex and its incident edges. Understanding and calculating γt is essential in analyzing and comprehending the structural properties of graphs, offering insights into their connectivity and dominance characteristics. This concept forms a fundamental aspect of graph theory, contributing to the exploration of diverse mathematical and computational applications.

Relation between Domination Number and Total Domination Number:

In graph theory, the link between the dominance number (γ) and total domination number (γt) is significant. [39]. Every time, the dominance number (γ) is either equal to or larger than the total domination number (γt). The formulations of these graph-theoretic ideas immediately lead to this connection. The total domination number in a graph takes into account both the dominating set and its neighbouring vertices, while the domination number in a graph indicates the smallest size of a dominating set. In order to analyse the coverage and efficiency of dominant sets in graph topologies and get insight into the resilience and connection of graphs, it is essential to comprehend this relationship.

Some basic result for domination and total domination

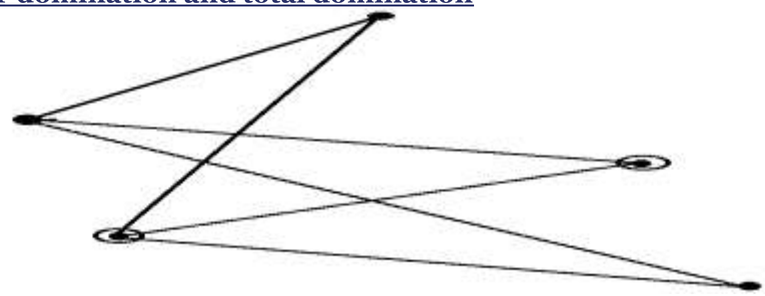


Figure No.4 Some basic result for domination and total domination

Total Domination in Graph Theory and its Theorems

A subfield of discrete mathematics called graph theory examines the connections between nodes, or vertices, in networks. The idea of dominance, in which some vertices govern or affect others, is one of the central ideas of graph theory. We explore the intricacies of vertex dominations in this thorough examination, concentrating on complete domination and its several theorems.

Theorem:1 $\gamma \leq n/2$
Let G be a graph where no vertex is isolated. The minimal size of a dominating set in G is known as the total dominance number, or γ. A basic conclusion is established by Theorem 1, which says that for each such graph, the total dominance number has a limit of 2.

Proof: Let D be a γ-set in graph theory, where γ is the total dominance number. Since there are no isolated vertices in graph G, every vertex in D needs to have at least one neighbour outside of D. As a result, a dominant set is also formed by the set V-D, which is made up of vertices that are not in D. On the other hand, choosing V-D as a dominant set goes against D's minimality as a γ-set if the cardinality of D (|D|) is greater than 2. It implies, then, that $\gamma = |D| \leq 2$. In networks lacking isolated vertices, this logic emphasises the connection between dominating sets and the overall domination number.[40].

Theorem:2 The presence of individual neighbours. According to this theorem, any graph G that has no isolated vertices has a γ-set D that has a private neighbour v' for each vertex v in D that is next to v but not to any other vertices in D.

Theorem:3 Triple Inequality for Total Domination/ Let G be a graph with no isolated vertices. Then $\gamma \leq \gamma t \leq 2\gamma$. For a graph G without isolated vertices, Theorem 2 states a triple inequality that relates the domination number (γ), total domination number (γt), and twice the domination number (2γ). This theorem offers a greater understanding of the structural properties of graphs without isolated vertices by shedding light on the connections between these important factors.

Proof: The definitions of dominance number and total domination number immediately lead to the inequality $\gamma \leq \gamma t$. Theorem 2.2.2 ensures that isolated vertices in the subgraph created by D have private neighbours when D is a γ-set but not a complete dominating set. After adding these private neighbours to D, a new set, D', with at most 2|D| vertices, is produced.[41]. Since D is a total dominating set, the second inequality—$\gamma t < 2|D| = 2\gamma$—is established since |D| must be bigger than or equal to γt. In the context of graph theory, this clarifies the complex link between dominance and ultimate domination..

Theorem:4 Lower Bound for Total Domination/ Let G be a graph of order n with no isolated vertices. Then $\gamma t \geq n/\Delta$ This theorem establishes a lower bound for the total domination number in terms of the maximum degree (Δ) of the graph.

Proof: γt-set S in a graph G, the definition dictates that every vertex in G is connected to at least one vertex in S, establishing the relationship N(S) = V(G). Given that each vertex in S has a maximum of Δ neighbors, the inequality $\Delta\gamma t \geq |V(G)| = n$ is evident, where n represents the number of vertices in G. Dividing both sides of the inequality by Δ leads to the derived result, $\gamma t \geq \Delta$. This inequality signifies the minimum cardinality of a γt-set in relation to the maximum degree (Δ) of the vertices in G, providing valuable insights into the connectivity and structure of the graph.

The theorems discussed here significantly enhance our comprehension of total domination within graph theory, shedding light on the intricacies of graphs devoid of isolated vertices. Delving deeper into these concepts opens avenues for applications in diverse fields such as network design, optimization strategies, and real-world problem-solving scenarios. Understanding total domination is pivotal in deciphering the underlying structures and properties of graphs, offering valuable insights that extend beyond theoretical realms. As we navigate through these theorems, their potential impact becomes evident, fostering connections between abstract mathematical concepts and practical applications in various domains. The exploration of total domination continues to unveil new possibilities for addressing complex challenges and refining solutions across different disciplines[42].

INDEPENDENT DOMINATION

Definition:

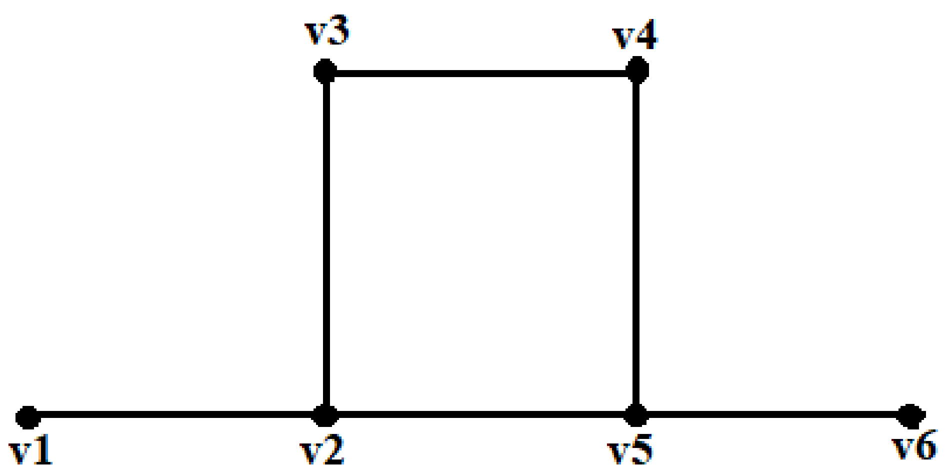
In graph theory, a set $S \subseteq V$ (where V denotes the vertex set of a graph G) is labeled as independent when none of its vertices are linked by an arc or edge. Specifically, a maximum independent set is characterized by the absence of any properly containing vertex set that maintains independence. This fundamental concept serves as the cornerstone for delving into the notion of independence within a graph. By identifying and analyzing maximum independent sets, graph theorists gain insights into the structure and connectivity of graphs, paving the way for a deeper understanding of relationships and patterns within these mathematical representations of interconnected entities.

Independent Domination Number:

One of the most important metrics in graph theory is the independent domination number (i(G), which is the lowest cardinality of a maximal independent dominating set in a graph G. Here, an independent dominating set (S) is a subset of G's vertex set with two essential characteristics.[43]: It is a dominant set, meaning that every vertex not in S is next to at least one vertex in S, and it is an independent set, indicating that no two vertices in S are adjacent. Determining i(G) is essential to studying the connectivity and structural characteristics of graphs since it provides information about how independent and dominant sets interact inside the graph structure.

γ(G)= {2, 4}

i(G)= {2, 4}
Therefore (G)= 2= i(G)



γ(G)= {v5, v2}

i(G)= {v3, v5, v1}
γ(G)=2 while i(G)=3
It is Clear that γ(G)≤i(G), for any graph G
γ(p5) = i(p5) for the route p5 (Fig A). Conversely, for the G graph in Fig. B, γ(G)=2 and i(G)=3. {v5, v2} really represents a γ-set for G, while {v3, v5, v1} represents a minimal independent dominant set of G.

Graph theory is a branch of mathematics that studies the relationships between various entities represented as vertices and the connections between them, represented as edges. One fundamental concept in graph theory is the notion of vertex domination, which plays a crucial role in understanding the structure and properties of graphs. In this comprehensive review, we delve into the theoretical aspects and practical applications of vertex dominations, focusing on theorems and corollaries that provide insights into the nature of these dominations.

Theorem:1

For any graph G, $i(G)+i(G') \le n-\Delta+\delta+1$.

Proof: The theorem in question sets an upper limit on the combined independent domination numbers of a graph G and its complement G′. Utilizing the relationship $\Delta i(G) \le n-\Delta$, where n represents the graph's order, the proof establishes that $i(G)+i(G') \le n-\Delta+\delta+1$. This inequality yields significant insights into the intricate dynamics involving independent dominations, vertex degrees, and minimum degrees within a graph. By connecting these elements, the theorem illuminates the structural aspects of graphs, shedding light on the constraints imposed by independent domination and offering a deeper understanding of the interplay between graph properties[44].

Theorem:2
If G is a k-regular graph (≥0k≥0), then i(G)≤n/2

Proof: This theorem focuses on the independent domination number within a regular graph. Consider a connected graph G, where i(G) denotes its independent domination number. The proof, under the assumption i(G) > n, presents a contradiction related to vertex degrees. Specifically, it argues that such a scenario contradicts the nature of regular graphs. Consequently, the conclusion is reached that i(G) ≤ n, imposing a restriction on the size of independent dominating sets within regular graphs. This result sheds light on the inherent structural limitations of regular graphs concerning independent domination, contributing to a deeper understanding of their properties and characteristics.

Theorem:3
For isolated-free graphs G and G′, ≤i(G)+i(G′) ≤n.

Proof: This theorem delves into the intricate connection between independent domination numbers within isolated-free graphs. Its proof meticulously differentiates between regular and non-regular cases, drawing upon insights gleaned from preceding theorems[45]. By doing so, it furnishes a comprehensive comprehension of the nuanced ways in which the structural attributes of a graph intricately shape its independent domination number. This exploration contributes to the broader field of graph theory, shedding light on the intricate interplay between graph structures and their independent domination characteristics. The theorem's analysis aids in unraveling the complexities associated with the dominance properties of graphs, enhancing our grasp of their inherent mathematical intricacies.

Corollary:
i(G)+i(G′) =n+1 implies either G or G′ is complete.

Proof: Corollary expanding on Theorem 3, this corollary presents a compelling outcome. It posits that when the sum of independent domination numbers reaches 1n+1, either graph G or its complement G′ must be a complete graph. The proof method involves examining isolated vertices and applying insights gleaned from preceding theorems. This result adds depth to graph theory, highlighting a significant connection between independent domination numbers and the completeness of graphs. Through a meticulous exploration of mathematical relationships, this corollary contributes to our understanding of graph structures and their inherent properties, enriching the field with valuable insights.

Theorem:4
In graph G, each maximum independence set is a minimum dominant sets.

Proof: This theorem illuminates a crucial link between maximal independent sets and minimal dominating sets within the realm of graph theory. It asserts that a maximal dominating set inherently possesses minimality, unraveling profound insights into the intricate interplay of these fundamental concepts. In graph theory, where structures and connections are paramount, this theorem contributes to a richer comprehension of the nuanced relationships between maximal independent sets and minimal dominating sets[46]. By showcasing the inherent minimality of maximal dominating sets, the theorem underscores the intricacies and dependencies inherent in the formation and characteristics of sets within graph structures, thereby enhancing the theoretical foundation of graph theory.

the theorems and corollaries presented in this review contribute significantly to the understanding of vertex dominations in graph theory. The results provide a theoretical framework for analyzing independent domination numbers and their relationships with graph properties. Moreover, these findings have practical implications in various applications, ranging from network design to optimization problems where efficient domination strategies are crucial. Further research and exploration in this field promise to unveil additional insights and applications, advancing the understanding of vertex dominations in graph theory.

CONNECTED DOMINATION (CDS)

Definition:
Connected Domination (CDS) is a vital concept in graph theory, playing a crucial role in understanding network structures and their resilience. In graph G= (V, E), a dominating set D is termed a connected dominating set if it induces a connected subgraph. The connected domination number, denoted as γ(G), represents the minimum cardinality of a connected dominating set in graph G.

Connected domination number:
The connected domination number holds significance in analyzing the efficiency and vulnerability of networks. A minimum connected dominating set (CDS) is one where its size equals the domination number. This subset

of vertices not only dominates the entire graph but also maintains connectivity, making it an essential parameter in various graph applications and network design scenarios[47].

An example of equality in domination, total domination, connected domination:

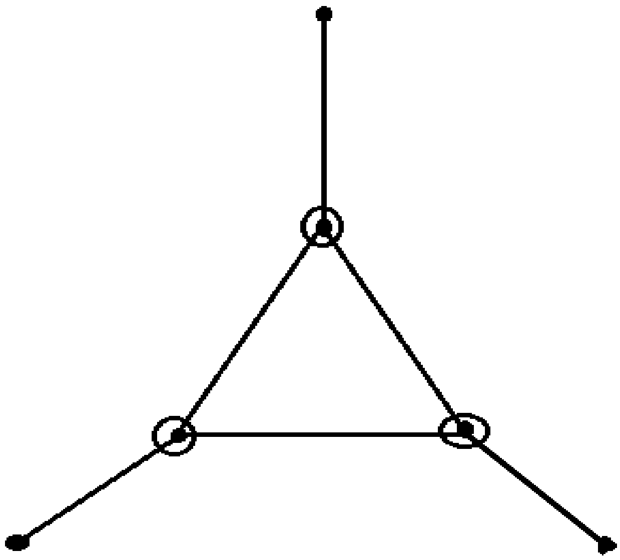


Figure No.5 Connected Domination(CDS)

Let l(G) denote the maximum leaf number of a graph Which is maximum number of leaves in a spanning tree. Connected Domination is a crucial concept in graph theory that plays a significant role in understanding the structure and connectivity of graphs. The concept involves the domination of a graph by a set of vertices, ensuring that every vertex in the graph is either part of the dominating set or adjacent to at least one dominating vertex. In this comprehensive review, we delve into the intricacies of Connected Domination, focusing on its theoretical foundations and practical applications.

Theorem:1 $\gamma c = n - \ell(G)$

Proof: The theorem establishes a fundamental relationship between the order of a graph (n), the connected domination number (γc), and the dominating number ($\ell(G)$). The proof begins by considering the case of a tree (T), where $\gamma c(T) = V(T) - \ell(T)$. Importantly, it is noted that a Connected Dominating Set (CDS) for a spanning tree T of G is also a CDS for G. This observation lays the groundwork for the subsequent argument.

$\gamma c \leq n - \ell(G)$

Next, let D be a minimum CDS of G and H be a spanning tree of the subgraph induced by D, denoted as G[D]. By connecting H to every vertex in V−D, a spanning tree of G is obtained. It is asserted that every vertex in V−D is a leaf of this spanning tree T. To understand this, consider the fact that if a vertex in V−D is not a leaf, removing it from the tree would still leave a connected dominating set, contradicting the minimality of D. Conversely, if x is a leaf of T, it implies that x is not in V−D. If x were in V−D, removing it from the tree would yield a CDS for the spanning tree T, and consequently, a CDS for G, again contradicting the minimality of D[48].

This proof establishes the equality $\gamma c = n - \ell(G)$, providing a clear and concise expression for the connected domination number in terms of the order and dominating number of a graph. The significance of this result extends to its applications in understanding the connectivity properties of graphs and identifying optimal strategies for establishing connected dominating sets. Connected Domination has diverse applications, ranging from network design to facility location problems. In network communication, identifying a minimal connected dominating set is crucial for efficient message propagation and fault tolerance. Moreover, in the context of facility location, the concept aids in determining optimal locations for facilities to ensure comprehensive coverage of a region[49].

Graph of G in proof of theorem given below:

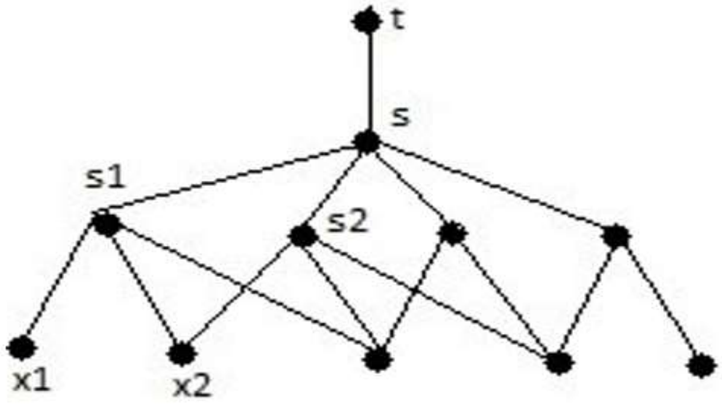


Figure No.6 Connected Domination in Complement Graphs

Theorem:2 Connected Domination in Complement Graphs

Consider a graph G of order n≥4, and let G' be its complement. If both G and G' are connected graphs, then the sum of the connected domination numbers of G and G' is bounded by n(n−3), as expressed in the inequality $\gamma c(G)+\gamma c(G') \leq n(n-3)$.

Proof of Theorem:2

We begin the proof by noting that a tree, a type of connected graph, has at least 2 leaves. Utilizing this fact, we can establish a lower bound for the connected domination number γc(G):

$\gamma c(G) \geq n-2$

Furthermore, since G is connected, we know that the number of edges in G, denoted as |E(G)|, satisfies |n−1≤|E(G)|. Substituting this into the previous inequality, we obtain:

$\gamma c(G) \geq 2(n-1) - n = 2|E(G)| - n$

A similar argument can be made for the connected domination number γc(G′) in the complement graph G':

$\gamma c(G') \geq 2|E(G')| - n$

Combining these inequalities, we can express the sum of connected domination numbers:

$\gamma c(G)+\gamma c(G') \geq 2(|E(G)|+|E(G')|) - 2n$

Simplifying further, we get:

$\gamma c(G)+\gamma c(G') \geq 2nC2=2n$

Thus, we have established the inequality:

$\gamma c(G)+\gamma c(G') \geq 2n$

To complete the proof, we compare this result with the upper bound n(n−3):

$\gamma c(G)+\gamma c(G') \leq n(n-3)$

Combining the upper and lower bounds, we arrive at the final conclusion:

$\gamma c(G)+\gamma c(G') \leq n(n-3)$

This completes the proof of Theorem 2.4.2, providing a valuable insight into the relationship between the connected domination numbers of a graph and its complement.

TOTAL VERTEX-EDGE DOMINATION

Let us assume that n = |V | is the order of the graph G = (V,E). N(v) = {u ∈ V | uv ∈ E} represents the open neighbourhood of a vertex v ∈ V, while N[v] = N(v) ∪ {v} represents the closed neighbouring. A vertex v's cardinality is its degree, degG(v), which may be written as follows. A vertex of degree one is referred to as a leaf vertex, while its neighbour is referred to as a support vertex. G[S] is the subgraph created by the vertices of a set S ⊆ V. The closed neighbourhood is denoted by N[S] = N(S) ∪ S, whereas the open neighbourhood is denoted by N(S) = ∪v∈SN(v). A vertex u ∈ V \ S that is next to v but not to any other vertex of S is called an S-external private neighbour of a vertex v ∈ S. The Ss-external private neighbour set of v, or the set of all S-external private neighbours of v ∈ S, is denoted by the notation epn (v, S). If every vertex in V − S has a neighbour in S, a complete dominating set is a subset S ⊆ V; conversely, a dominating set of G is a subset V that contains every vertex in S. The lowest cardinality domination number, γ(G) (or total domination number, γt(G)), corresponds to a dominating set (or total dominating set) of G. A \t(G)-set is a minimum cardinally complete dominating set of G. Cockayne, Dawes, and Hedetniemi presented total dominance first.[50].

$D \subseteq V$ is considered independent if it has no two neighbouring vertices. Every set $D \subseteq V$ is a packing set of G if D is independent and none of its vertices share a common neighbour; that is, $N[x] \cap N[y] = \emptyset$ holds for any two different vertices $x, y \in D$. Put otherwise, every edge incident to a vertex in N[v] and every edge next to these incident edges are dominated by a vertex v. Every edge uv consequent to v is subordinate to a vertex v. A set $S \subseteq V$ is a vertex-edge dominating set (or simply a ve-dominating set) if there is a vertex $v \in S$ such that v ve-dominates e for every edge $e \in E$. A set $S \subseteq V$ is an independent vertex-edge dominating set (or merely an autonomous ve-dominant set) if it is also independent and ve-dominant. The majority number \ve(G) at the vertex edge represents a ve-dominating set of G's shortest cardinality, while ive(G), the independent vertex edge domination number, represents an independent dominating set of G's lowest a cardinality Peters [8] first proposed the idea of vertex-edge dominance in his 1986 PhD a dissertation, and it was subsequently studied.

We demonstrate and examine the whole vertex-edge dominance in this paper. An entire A subset $S \subseteq V$ of G is the vertex-edge dominating set, or simply the complete ve-dominating set. Every vertex in S has a neighbour in S if S is a ve-dominating set, and the subgraph it creates is free of isolated vertices. A whole ve-dominating set's least cardinality is equal to γt ve(G), which is G's entire vertex-edge domination number. In this paper, we only analyse nontrivial connected graphs, or ntc graphs. First, we prove that the problem of determining the total ve-domination number is NP-complete, even if restricted to bipartite graphs. Next, we impose a few upper bounds on a graph's overall ve-domination number. More precisely, we demonstrate that if T is a tree that is not a star with order n, leaves, and s support vertices, then γt ve(T) < (n − + s)/2. There is also a description of the trees that go above this upper limit. Moreover, it is noted that, generally speaking, the two natural upper bounds for γt ve (G), γt(G), and 2γve(G) cannot be compared. In conclusion, we provide an adequate prerequisite that guarantees that, for graphs G, γt ve (G)=2ηve (G). Additionally, we describe any tree T that has ηt ve (T)=2ηve (T)[51]

COMPLEXITY RESULT

In this part, we will examine the intricacy of the subsequent decision-making task, which we will designate as ENTIRE DOMINANCE:
Total ve-Dom (Total ve-DOMINATION) = 2.
Graph G = (V,E) using k, a positive integer, < |V |, as an example.
Is there, at most, a ve-dominating set of cardinality for G?
By simplifying the well-known NP-complete problem Exact-3-Cover (X3C) to Total ve-Dom, we demonstrate the NP-completeness of this issue.
A finite set X with |X| = 3q and a collection C of 3-element subsets of X are the examples of an EXACT 3-COVER (X3C).
Is it possible for every element of X to appear in precisely one element of C in a subcollection C of C?
Theorem 1: Issue NP-Completeness of total ve-Dom holds for bipartite graphs.
Proof. Since we can demonstrate in polynomial time that a set with cardinality at most k is a total ve-dominating set, total ve-Dom is a member of N P. Let us now demonstrate how to transform any instance of X3C into an instance G of Total ve-Dom such that only one instance will have a solution, even if both instances have them. Let $C = \{C_1, C_2,..., C_t\}$ be an arbitrary instance of X3C. Let X be = $\{x_1, x_2,..., x_{3q}\}$. The path we create for each $x_i \in X$ is $P_2 = x_i$-y_i. Let F be $\{x_1, x_2,..., x_{3q}\}$ and let W be the set of all edges x_iy_i..

For each $C_j \in C$, we build a route of order 5, P_j 5, which is u_j -v_j -w_j -z_j -c_j. Let Y be defined as $\{c_1, c_2,..., c_t\}$. We now add edges c_jx_i to a graph G if $x_i \in C_j$. It is clear that G is a bipartite graph. Assume that each Vs produces a subgraph of G, H (P_j 5). Put k in charge of 2t+q. Because any complete ve-dominating set D of G has at least vertices from each route, take note that $|D \cap V(H)| \geq 2t$. P_j 5..

Let us assume that there exists a solution C for the X, C instance of X3C. In this way, we can create a whole prevailing set D of G with weight k. For each P_j 5, enter w_j and z_j in D. Additionally, insert c_j in D for each $C_j \in$ C. Observe that C has specifically q the cardinality since it exists. This means that $\{x_1, x_2,..., x_{3q}\}$ contains q c_j's with disjoint neighbourhoods. Since C is a solution for X3C, every edge incident with a vertex of W is ve-dominated by some vertex c_i. Furthermore, every vertex in D has a neighbour in D, and D ve-dominates all of H's edges. As a result, D is the whole ve-dominating set of G and has size 2t + q = k.

Conversely, suppose that G has a complete ve-dominating set D, with a maximum size of k. We may suppose that D does not include the vertex y_i, thus x_i can be used in its place without sacrificing generality. Because $|D \cap V(H)| \geq 2t$, $|D \cap W| \leq q$ as a consequence. The fact that each vertex of $W \cap D$ can only ve-dominate one edge of F and that |F| = 3q lead to $D \cap Y = \emptyset$ as well. Suppose $r = |D \cap Y|$. The fact that r <= q is obvious. $D \cap Y$ ve-dominates at most 3r edges from F, as every c_j in W has exactly three neighbours. As a result, $|D \cap W| \geq 3q - 3r$. We now arrive at $r \geq q$ using the information that $|D| \leq k = 2t + q$ and $|D| = |D \cap V(H)| + |D \cap W| \geq 2t + r + 3q - 3r$. Consequently, $|D \cap W| = 0$ because r = q. C = $\{C_j: c_j \in D\}$ is hence an ideal cover for C.

VERTEX SEQUENCES IN GRAPHS

kinds of vertex sequences, each of which is characterised by a condition that the subsequent vertex in the series has to satisfy. We specifically take into account needs related to dominance and associated graph characteristics. We describe four categories of sequences that have been characterised as "dominating" in the literature in this paragraph. Then, we concentrate on the dominant sequence and the entire dominating sequence, two of these sequences. We provide a quick summary and some fresh findings on dominant sequences.[52].
First, we define certain terms.

Assume that the graph G = (V, E) has size |E| and order |V |. Let S = (v1, v2,..., vk) be an ordered sequence of unique vertices, and let Sb = {v1, v2,..., vk} be a compatible set of vertex. Sbj = {vi | 1 ≤ i ≤ j.} is the collection of vertices in the first j positions in S; this will make things simpler to grasp.

The open neighbourhood of a vertex v ∈ V is the set N(v) = {u | uv ∈ E}, whose vertices are called neighbours of v. The closed neighbourhood of vertex v is given by the set N[v] = N(v) ∧ {v}. The number of v's neighbours, or |N(v)|, equals the degree d(v) of v. G's vertices are arranged with δ(G) representing the least degree and ∆(G) representing the maximum degree. The set N[S] = N(S) ∧ S represents the closed neighbourhood of a set S, whereas the set N(S) = S v∈SN N(v) represents the open neighbourhood.

The set pn [v, S] = N[v]\ (N [S \ {v}]) represents the S-private neighbourhood of a vertex v ∈ S; vertices in this set are called private neighbours of v (with regard to S). If there are no neighbouring vertices in a set S of vertices, then the set is independent. The greatest cardinality of an independent set in (G) is known as the vertex independence number, or α(G).

If every vertex in V \ S = S has a neighbour in S, or N[S] = V, then set S is a dominant set. If N(S) = V, then every vertex in V has a neighbour in S, making set S a completely dominating set. If there isn't a big enough subset of S to qualify as a (total) dominating set of G, then a (total) dominating set S of G is minimal. The lowest cardinality of a dominating set in G is indicated by the domination number γ(G), while the highest cardinality of a minimal dominating set is shown by the higher domination number Γ(G). The meanings of the total dominance number γt(G) and the higher total domination number Γt(G) are comparable. Let G[S] be the subgraph generated by the set S ⊆ V in G. A maximum complete subgraph of a graph G is called a clique, and a clique with k vertices is known as a k-clique.[53].

1. NEIGHBORHOOD VERTEX (DOMINATING) SEQUENCES

sequences determined by the vertices' open and closed neighbourhoods. These kinds of sequences may be broadly classified into four categories: (i) closed neighbourhood, (ii) open neighbourhood, (iii) closed-open neighbourhood, and (iv) open-closed neighbourhood.

a) Closed neighborhood

A closed neighbourhood sequence S = (v1, v2,..., vk) of different vertices is if and only if for any i where 2 ≤ i < k,

$$N[v_i] \not\subseteq \bigcup_{j=1}^{i-1} N[v_j].$$

A vertex vi is the dominant vertex in its closed neighbourhood N[vi], meaning that it is the dominant vertex in all of its neighbours as well as itself. This kind of sequence occurs when a vertex vi dominates at least one vertex x that isn't dominated by any of the sequence's previous vertices, therefore for i ∈ {2, 3,... k}

$$N[v_i] \setminus N[\widehat{S}_{i-1}] \neq \emptyset.$$

Keep in mind that this vertex x might be the vertex vi itself. Another way to put it is that each vertex vi has to dominate at least one vertex that wasn't previously dominated. Put otherwise, pn [vi, Sbi] 6= ∅.
Sb has to be a dominant set of G if S is a closed neighbourhood sequence of maximum length in G. As a result, S is referred to as G's dominant sequence. The Grundy dominance number of G, represented as γgr(G), is the maximum length of a dominating sequence S of a graph G, or |Sb|. A Grundy dominating sequence, or simply a γgr-sequence of G, is a dominating sequence of length γgr(G).

Proposition 2.1. The domination number γ(G) is equal to the minimal length of a maximum dominating sequence for every graph G.

Bresar et al. credit [] for introducing the dominance game as the inspiration for their investigation of these sequences. In this game, In order to grow the set of vertices that G has controlled up to At that moment in the game, the players Dominator and Staller alternately choose one of the two Vertices. Notwithstanding the opposing objectives of the two players (Dominator seeks to minimise the number of moves in the game, while Staller seeks to maximise the number of moves), the outcome of the game is a sequence of vertices with the property that each chosen vertex dominates at least one previously dominated vertex. The game ends when either all possible moves have been made or when the vertex sequence created by those moves becomes a dominating sequence of G. It is evident that the optimal outcome for Dominator is a dominant sequence of length γ(G), whereas the optimal outcome for Staller is a dominating sequence of length γgr(G), also known as a Grundy dominating sequence. Naturally, in the dominance game, the dominating sequence lengths might deviate from these two extremes.[54].

b) Open neighborhood

If for every i with 2 ≤ i < k, the sequence S = (v1, v2,..., vk) of different vertices is an open neighbourhood sequence,

$$N(v_i) \not\subseteq \bigcup_{j=1}^{i-1} N(v_j).$$

A vertex v total (open) does not total dominate itself; rather, v total dominates every other vertex in its open neighbourhood N(v). Therefore, a sequence S = (v1, v2,..., vk) is an open neighbourhood sequence for any graph G without isolated vertices if and only if for each vertex vi where i ∈ {2, 3,... k},

$$N(v_i) \setminus N(\widehat{S}_{i-1}) \neq \emptyset.$$

Since no vertex in the sequence before it is total dominated, each vertex vi in the series entire thus controls at least one vertex. Stated otherwise, vi is next to a minimum of one neighborless vertex in Sbi−1. S is referred to be a total dominating sequence if Sb is a total dominating set of G. The Grundy total domination number, γ t gr(G), is the maximum length of a total dominating sequence of G. A total dominating sequence of length γ t gr(G) is known as a Grundy total dominating sequence, or γ t gr-sequence of G. The authors Bresar, Henning, and Rall (2016) established the idea of complete dominant sequences. They observed that the total domination number, γt(G), is equal to the shortest length of a graph G's total dominating sequence. Complete dominating episodes and the entirety of dominance are connected in the same way tha.[55].

c) Closed-open neighborhood

For any i where 2 ≤ i < k, the sequence S = (v1, v2,..., vk) of different vertices is a closed-open neighbourhood sequence.,

$$N[v_i] \not\subseteq \bigcup_{j=1}^{i-1} N(v_j).$$

Accordingly, for every vertex vi, for i ∈ {2, 3,... k}, a sequence S = (v1, v2,..., vk) in a graph G is termed a closed-open neighbourhood sequence.,

$$N[v_i] \setminus N(\widehat{S}_{i-1}) \neq \emptyset.$$

Stated differently, every vertex in N[vi] has an empty neighbour in N(Sbi−1). S is a closed-open dominating sequence if Sb is a dominating set. The term "L-sequence" refers to a closed-open dominant sequence that is closely related to a certain kind of zero forcing number (the definition of zero forcing is provided in Section 2.5). The L-Grundy dominance number of G is the length of the longest L-sequence and is represented by γ L gr(G).

d) Open-closed neighborhood

If for every i with 2 ≤ i ≤ k, the sequence S = (v1, v2,..., vk) of different vertices is an open-closed neighbourhood sequence.,

$$N(v_i) \not\subseteq \bigcup_{j=1}^{i-1} N[v_j].$$

Therefore, for every vertex vi, for i ∈ {2, 3,... k, a sequence S = (v1, v2,..., vk) in a graph G is termed an open-closed neighbourhood sequence.

$$N(v_i) \setminus N[\widehat{S}_{i-1}] \neq \emptyset.$$

Stated otherwise, at least one vertex that is not in N[Sbi−1] is next to vi. Once again, S is an open-closed dominating sequence if Sb Sb is a powerful set. Because of their strong correlation with the zero forcing number, open-closed dominant sequences are also known as Z-sequences, as we shall see in Section 2.5. The Z-Grundy dominance number of G, denoted as γ Z gr(G), is the length of the longest Z-sequence.

APPLICATION OF DOMINATION IN GRAPH

The utilisation of domination in graph theory extends to various real-world scenarios, particularly in optimizing resource allocation and minimizing costs in different fields. One notable application is Despite issues with facility placement, where the objective is to strategically place establishments like fire stations or hospitals to minimize travel distances for individuals. This involves identifying dominating sets of locations that ensure efficient coverage and accessibility. Concerning facility placement issues, domination helps in figuring out the bare minimal requirements for facilities required to serve a population effectively. Alternatively, it addresses scenarios where the maximum distance a person should travel to reach a facility is fixed, and the goal is to minimize the number of facilities needed. This application finds relevance in urban planning, emergency services deployment, and infrastructure development. Another application of domination in graph theory is in the context of finding sets of representatives. In various scenarios, it is essential to identify a subset of elements that represent the entire set in terms of certain properties or characteristics. Domination concepts assist in efficiently selecting such representative sets, contributing to tasks like decision-making, sampling, or data analysis[56].
Additionally, domination has implications in monitoring communication or electrical networks. Identifying dominating sets in these networks ensures effective surveillance and control, helping in the detection of faults or irregularities. Furthermore, in land surveying, domination concepts are applied to minimize the number of surveyor positions required to measure the height of a whole area, leading to cost-effective and efficient surveying practices. The application dominance in graph theory spans various domains, from facility location and representative set identification to network monitoring and land surveying. These applications demonstrate the practical significance of theoretical concepts in solving real-world optimization problems.

SCHOOL BUS ROUTING

In the realm of graph theory, the application of vertex domination finds practical significance in various real-world scenarios. One such application involves optimizing school bus routes for the efficient transportation of students. The objective is to design routes that minimize the walking distance for each child to reach the bus pickup point, ensuring accessibility within a specified range. Consider a scenario where a school aims to establish an effective transportation system adhering to certain constraints. The city's street map is shown as a graph, with vertices corresponding to to pick up blocks and edges denoting the routes. The school's location is indicated by a large vertex. To streamline the bus routes, the concept of vertex domination comes into play. In this context, the school wishes to ensure that no child has to walk more than a predetermined distance, such as two blocks, to reach a bus pickup point. This constraint is crucial for the safety and convenience of the students[57]. The task involves constructing optimal routes for school buses, ensuring that every child is within the specified distance from a pickup point. Moreover, additional constraints may include limits on the duration of bus rides and the maximum number of children a bus can accommodate simultaneously. By leveraging vertex domination strategies in graph theory, the school can systematically plan and optimize bus routes, thereby enhancing the efficiency of student transportation while prioritizing safety and convenience. This application showcases the practical implications of theoretical concepts in graph theory within the context of real-world problem-solving.

Figure No.7 School bus routing

LOCATING RADAR STATIONS PROBLEM

In the context of graph theory, the application of domination finds relevance in addressing complex real-world problems, such as the Locating Radar Stations Problem. This particular conundrum, extensively examined by mathematician Berge, involves the strategic surveillance of numerous locations[58]. The primary objective is to minimize the number of radar stations required for effective monitoring. The problem revolves around identifying a subset of strategic locations where radar stations can be strategically placed to ensure comprehensive surveillance. The challenge is to determine the optimal set of locations that would enable the radar stations to cover the entire area while minimizing redundancy and operational costs. Graph theory provides a powerful foundation for describing and resolving these kinds of optimisation issues. The locations are shown as nodes in this scenario a graph, and the connections between them as edges. The concept of domination comes into play as the goal is to find a subset of nodes (dominating set) that covers all other nodes in the graph. By identifying the minimum dominating set, mathematicians and researchers aim to propose efficient solutions for the placement of radar stations, contributing to the field of optimization and strategic surveillance.

NUCLEAR POWER PLANTS PROBLEM

The application of domination in graph theory is exemplified by the nuclear power plants problem. This problem involves determining the optimal placement of guards to watch out for warning lights at different places. If a guard at position x may see the warning light at position y, after which an arc is drawn connecting that position and the other y in this situation. The most important query is: How many guards are required to properly monitor every warning light, and where should they be positioned strategically? In order to solve such practical issues, graph theory's core idea of dominance comes in rather handy.[59]. Domination is essentially about picking a subset of nodes in the graph to perform a certain function, and making sure that every node in the network is close to at least just one subset node. Due to its many applications in a variety of fields, such as distributed computing, social networks, biological networks, ad hoc networks, and web graphs, this idea has drawn more attention [1, 25, 27, 47].

The significance of domination in these applications lies in its ability to optimize resource allocation and enhance the efficiency of network-related tasks. By strategically identifying and placing dominating nodes, the overall connectivity and coverage of the network can be improved. The nuclear power plants problem serves as a concrete example of how the concept of domination can be applied to model and solve complex real-life problems, ensuring the effective surveillance of warning lights and the security of critical infrastructure. As graph theory continues to evolve, the applications of domination are likely to expand, offering innovative solutions to a wide range of network-related challenges[60].

MODELING BIOLOGICAL NETWORKS

Graph theory, with its versatile applications, finds a significant role in modeling biological networks, particularly in the context of RNA structures. The utilization of graph theory provides a valuable approach to understanding and analyzing complex biological systems. In this context, one of the key graph invariants employed is the domination number, which plays a crucial role in identifying secondary RNA motifs. RNA, or Ribonucleic acid, is a fundamental molecule in biology, and its secondary structure, represented as trees in graph theory, holds valuable information. The domination The lowest number in a graph is called its number. of nodes that need to be occupied or controlled to exert influence over the entire graph. In the realm of RNA structure analysis, variations in the domination number become essential indicators.

Research in this domain has demonstrated that studying the differences in the dominance number allows for a nuanced differentiation between trees representing native RNA domains as well as others which are more likely to be RNA. This numerical approach enables researchers to discern subtle distinctions within the structural motifs of RNA, providing insights into the intricate relationships and functionalities of these biological networks. By employing graph theory and domination number variations, scientists and researchers can enhance their ability to identify and characterize secondary RNA motifs accurately[61]. This methodology contributes to a deeper understanding of the structural diversity in biological networks, opening doors for advances in drug discovery, molecular biology, and bioinformatics. All things considered, the use of dominance in graph theory helps to simplify the intricate molecular workings of biological systems.

MODELING SOCIAL NETWORKS

Modeling social networks through the application of domination in graph theory provides a valuable framework for understanding the dynamics of relationships within a community. Social networks, composed of individuals or groups interconnected by various types of dependencies, are complex structures that can be effectively analyzed using mathematical concepts, specifically dominating sets in graphs. The theory of social networks involves identifying target individuals or groups within the network, a task that is crucial for various applications. Kelleher and Cozzens delved into this area, demonstrating that graph theory can be employed to model social networks. Graph theory, with its nodes and edges representing individuals and their connections, respectively, allows for a systematic analysis of the relationships within a social network[62].

An essential idea in graph theory, dominating sets, are crucial to this modeling process. These sets consist of nodes that exert control or influence over the entire network, showcasing their significance in understanding the overall dynamics. Identifying dominating sets aids in pinpointing key individuals whose actions or decisions have a substantial impact on the network[63]. Kelleher and Cozzens' work highlights that properties of prevailing groups in graphs can be harnessed to identify and analyze sets of individuals within social networks. This not only contributes to a better comprehension of social structures but also has practical implications in fields such as sociology, psychology, and marketing, where understanding the influence and dynamics of important people is essential. Graph theory's use of dominance offers a potent tool for social network modelling and analysis. Through the use of dominant set features, scholars may get valuable understanding of the significant nodes in a network. contributing to a more profound understanding of the intricate dynamics inherent in social structures.

FACILITY LOCATION PROBLEM

The application of domination in graph theory finds significant relevance in addressing complex problems such as the Facility Location Problem (FLP) within operational research. Dominating sets in graphs serve as intuitive models for optimizing the allocation of facilities to enhance efficiency and achieve specific objectives. In the context of FLP, the primary concern is the strategic placement of one or more facilities to optimize a defined objective. The objectives in facility location problems often revolve around minimizing transportation costs, ensuring distributing services to clients fairly and gaining the biggest market share. By employing domination concepts in graph theory, analysts can identify sets of critical locations or nodes that efficiently cover the entire network. These dominating sets play a pivotal role in decision-making processes related to facility placement, as they contribute to the overall optimization of the system[64]. Graph theory, with its ability to represent and analyze relationships between interconnected elements, provides a powerful framework for tackling facility location challenges. The utilization of dominating sets not only aids in addressing optimization goals but also facilitates a comprehensive understanding of spatial relationships and resource allocation within the operational landscape. As a result, the application of domination in graph theory emerges as a valuable tool for enhancing decision-making processes in facility location problems.

COMPUTER COMMUNICATION NETWORK

In the realm of computer communication networks, the application of domination in graph theory plays a crucial role in optimizing information collection processes. A graph may be a useful model for the network., denoted as G = (V, E), where vertices (V) represent individual computers or processors, and edges (E) symbolize the direct links between pairs of computers. Consider a scenario where there are 16 computers forming a network, and the objective is to collect information from all processors periodically[65]. To achieve efficient information gathering, a concept known as dominating sets comes into play. A dominating set is a subset of vertices where each vertex that is not part of the set is adjacent to at least one of the set's members. Within the framework of computer networks, the goal is to identify a small set of processors that can efficiently collect information from all others. This set is referred to as a dominating set, and it ensures that information can be routed quickly without traversing overly long paths.

In the described scenario, the focus is on a specific type of dominating set known as a distance-2 dominating set. This entails selecting a set of processors that are in close proximity to one other, facilitating quick information exchange. The requirement is to accept a maximum two-unit latency between the time information is sent by a processor and when it gets to a collector in the vicinity. The two coloured vertices in the network's graphical representation create a distance-2 dominant set in the hypercube network.. This set fulfills the criteria of being close to all other processors and ensuring a rapid information collection process. The application of domination in graph theory thus proves instrumental in optimizing the efficiency of computer communication networks, particularly in scenarios where timely information collection is imperative[66].

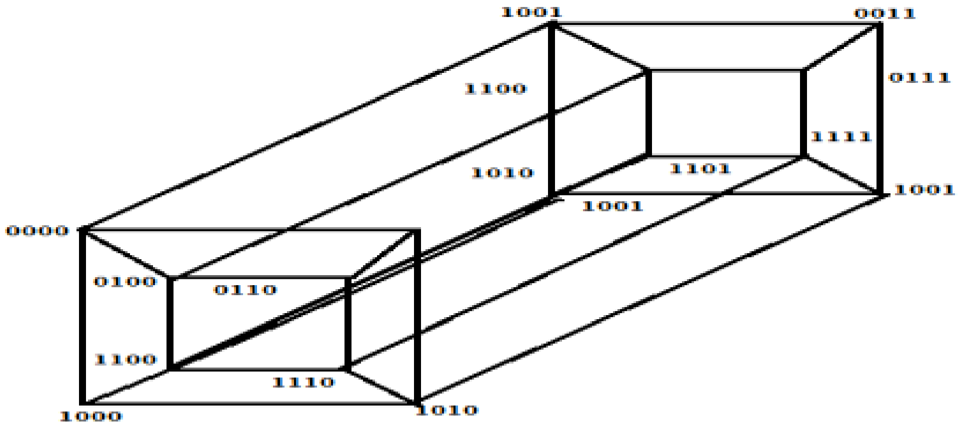


Figure No.8 Computer communication network

RADIO STATIONS

In radio station placement in remote villages, the application of domination in graph theory proves to be a valuable tool for optimizing resource allocation. In this scenario, each village is represented as a vertex in a graph, and edges between vertices are labeled with the distances between the corresponding villages. The objective is to strategically position radio stations in such a way that It is possible to effectively broadcast messages to every village within the region. The difficulty is to reduce the number of stations while maintaining coverage for every village because of each station's constrained broadcasting range and the corresponding expense.[67]. This problem aligns with the idea for dominance is graph theory, where a dominating set of vertices is sought to cover the entire graph. In the context of the radio station application, a dominating set would represent the villages where radio stations are placed to ensure communication with every other village. By employing domination techniques, one can analyze the graph structure and identify an optimal placement of radio stations. This not only minimizes costs but also maximizes the efficiency of message dissemination across the region. The application of domination in this context demonstrates the practicality of graph theory in solving real-world problems related to resource optimization and communication network design.

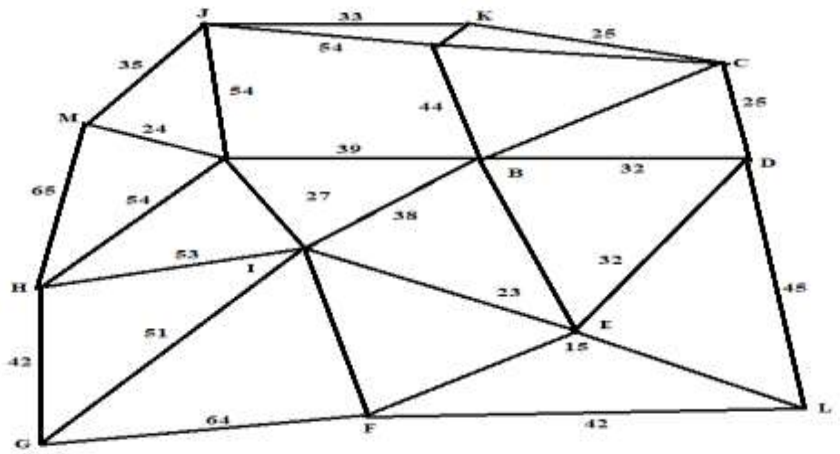


Figure No.9 Distance in Kilometer

The application of domination is evident in scenarios like a radio station network. Consider a graph with broadcast ranges indicated by edges and locations represented by vertices. Finding the bare minimum of stations needed to dominate every vertex within a 50-kilometer radius of a radio station is crucial. In Figure 2, a set {B, F, H, J} with a cardinality of four is identified. This set effectively dominates all other vertices within the 50-kilometer limit, showcasing the practical application of domination in optimizing radio station placement for efficient coverage in the given graph[68].

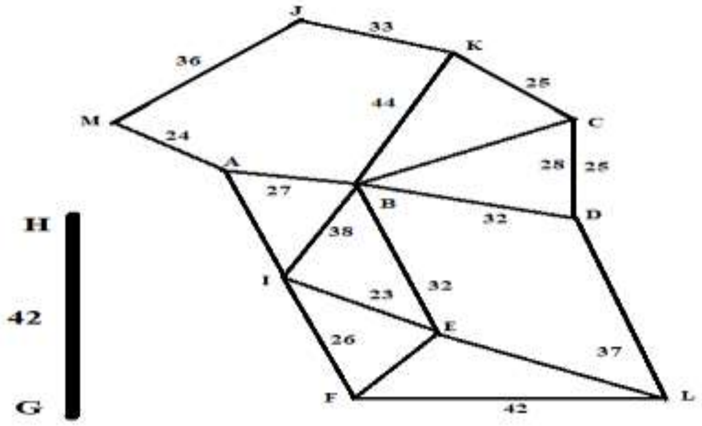


Figure No.10 Distance in Kilometer

The application of domination finds relevance in scenarios such as radio station coverage. Assuming a broadcast range of fifty kilometers, edges representing distances beyond this limit can be eliminated from the graph. The objective then becomes identifying a dominating set within this constrained graph. Notably, if the budget allows for radio stations with a seventy-kilometer broadcast range, the number required reduces to three stations[69]. This application showcases how domination concepts in graph theory can be practically employed to optimize resource allocation and coverage efficiency in real-world scenarios, illustrating the versatility of graph theory in addressing practical problems.

VERTEX DOMINATION OF GENERALIZED PETERSEN GRAPHS

For instance, we recommend that the reader study a chart theoretical book in order to understand the relevance of fundamental ideas that are not covered below. You refer to the viewer a number of studies that address the graph theory's idea of predominance. If every vertex in V – D lies next to a minimum of one vertex in D, then a set D of edges of a graph G is a (vertex) dominant set. The dimension of a minimal ruling set of G is the (vertex) dominance amount of G, represented as γ (G). A γ -set is a minimal dominant set of G. If each vertices in set D is controlled by precisely one vertices in set G, then set D is an efficient dominant set or flawless dominant set. Keep in mind that there has to be a separate group of effective dominants. Furthermore, every graph's efficient dominant set has to be of size γ (G). Let P (n, k) be an extended Peterson structure. Let the perimeter group equal {uiui+1, uivi, vivi+k}, 1 < i ≤ n, and let its vertex set be the union of U = {u1, u2,..., un} and V = {v1, v2,..., vn}. U-vertices make up the first set of vertices in whilst v-vertices make up the remaining class. If every vertex on a path in P (n, k) is a u-vertice, this path is referred to as a u-path. That is also how a v-path is defined. The border of the uivi is shown in the spoke. The generalised Petersen graph P(16, 5) and a powerful dominant set are shown in Fig. 11. Additional important factors for universality Foster graphs were studied by George's, Zelinka, and others. Here, we examine their control over edges. They discuss extended Christensen diagrams having optimal dominating sets in Chapter 2. This conclusion helps us determine the precise values of 'A (P(n, k)) in Section 3 for 1 < k = 3. In Moving on, γ (P(n, k)) is evaluated on each.[70].

Efficient vertex domination

A helpful required condition for P (n, k) to have an effective dominating set is provided in the following lemma..Lemma 1. n and 4|n = γ (P(n, k)) if P (n, k) contains an efficient dominating set

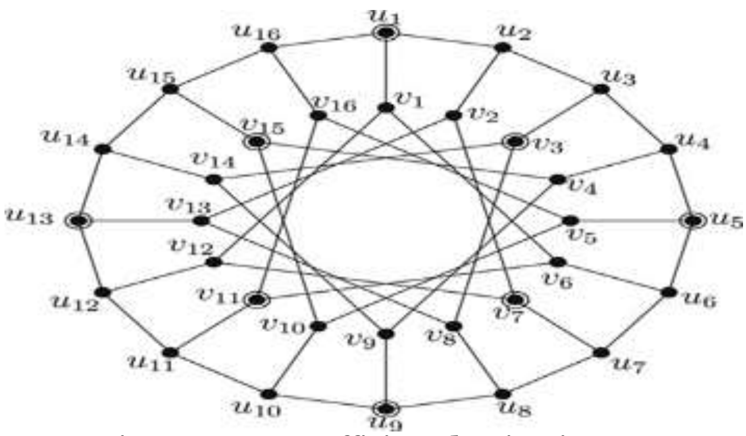


Figure No.11 An efficient dominating set

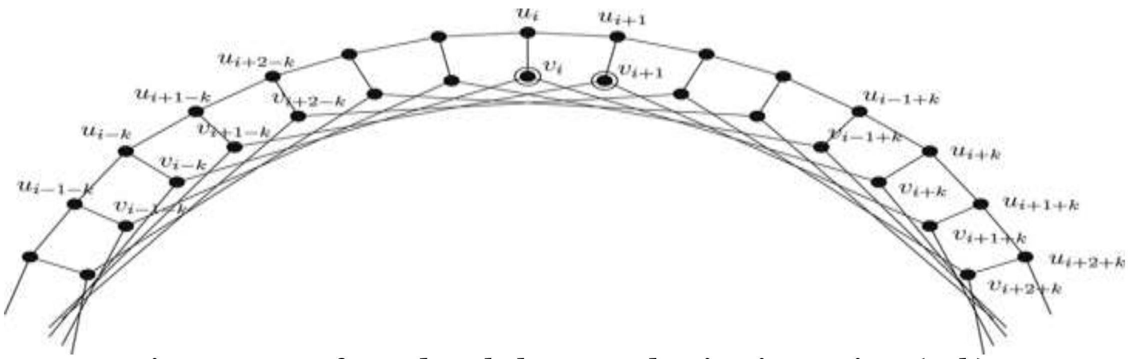


Figure No.12 If v_i and v_{i+1} belong to a dominating set in P (n, k).

Coding theory

In The use of dominance in coding theory is explained by Kalbfleisch, Stanton, Horton, and Cockayne., and Hedetniemi. By defining a graph where vertices represent n-dimensional vectors with coordinates from 1 to p (where p > 1), adjacency is established between vertices differing in only one coordinate[71]. This graph exhibits dominating sets with specific properties, serving as covering sets (n, p), perfect covering sets, or single error correcting codes. These sets are essential to the theory of coding., contributing to the design of error-correcting codes and covering sets with applications in reliable data transmission and storage. The incorporation of domination concepts enhances the understanding and utilization of graph theory principles in the realm of coding theory, facilitating advancements in efficient and robust communication systems.

MULTIPLE DOMINATION PROBLEM

Multiple domination is a crucial concept with diverse applications, particularly in the realm of computer networks. Its significance lies in constructing hierarchical overlay networks for efficient index searching in peer-to-peer applications. These overlay networks often serve as decentralised databases, enabling index searches in modern instant messaging and file-sharing networks. Finding a balance between fault tolerance and efficiency requires a variety of dominant sets. Furthermore, the distributed construction of minimal spanning trees depends on multiple dominance., optimizing network structures. In the dynamic landscape of modern computer networks, wireless sensor networks exemplify a direct and rapidly evolving application of multiple domination. This versatile concept proves invaluable in enhancing the functionality and performance of diverse network applications, contributing to the advancement of computational systems[72].

CONCLUSION

The comprehensive review on vertex dominations in graph theory brings to light the multifaceted nature of domination concepts and their wide-ranging applications. The thorough exploration of domination number and its variations demonstrates their relevance in protecting vertices and ensuring the stability of networks. With over 75 identified variations, the paper showcases the extensive research landscape within the field, offering a nuanced understanding of graph theory. The incorporation of additional conditions on subsets adds a layer of complexity, enriching the theoretical framework. The practical applications discussed in the paper underscore the real-world utility of graph theory, emphasizing its role in solving complex problems in science and engineering. The versatility of domination concepts is evident in their adaptability to various scenarios, making them invaluable tools for addressing challenges in diverse domains. The focus on specific areas such as planar graphs, connected graphs, and inverse dominations further illustrates the depth of research and the broad spectrum of applications. The project not only achieves its goal of elucidating the significance of graph theory but also contributes to the ongoing evolution of the field. Researchers in graph theory will find the paper to be a comprehensive and insightful resource, providing valuable information and ideas for further exploration. The paper's success lies in its ability to bridge theoretical concepts with practical applications, making it an essential reference for anyone interested in the dynamic world of graph theory. Overall, this comprehensive review serves as a testament to the enduring importance and applicability of vertex dominations in advancing the understanding of complex networks and systems.

FUTURE SCOPE

The future scope of the study on vertex dominations in graph theory holds immense potential for further exploration and application. As technological advancements continue to reshape various domains, understanding and leveraging vertex dominations can play a pivotal role in solving real-world problems. Future research can delve into developing advanced algorithms and computational methods for efficiently identifying and utilizing vertex dominations in large-scale networks, such as social networks, transportation systems, and communication networks. Moreover, exploring the theoretical aspects of vertex dominations in different types of graphs and extending the study to dynamic graphs and evolving networks can open new avenues for research. The application of vertex dominations in optimization problems, network design, and resource allocation presents promising directions for practical implementations. Collaborations with experts in related domains like operations research, computer science, and engineering, can foster interdisciplinary research, leading to innovative solutions and applications in diverse industries. As the world becomes increasingly interconnected, a comprehensive review on vertex dominations can serve as a foundation for addressing emerging challenges and shaping the future landscape of graph theory applications.

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