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Q-Pearson Differential Equation With Applications

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ARTICLE INFO **ABSTRACT**

Abstract. The Karl Pearson differential equation stands as a cornerstone in classical statistics, yielding pivotal distributions that underpin statistical analysis. This paper presents a significant extension of the Pearson equation, embracing the paradigm of nonextensive statistics and harnessing the power of the q-logarithm. Through this novel approach, a family of previously unexplored q-distributions emerges, demonstrating the profound interplay between classical and nonextensive statistical concepts. The practical implications of these newfound distributions are highlighted through their application to real-world datasets, which undergo rigorous scrutiny using both the Akaike and small sample Akaike information criteria. This comprehensive analysis underscores the versatility and effectiveness of the proposed framework, fostering a deeper understanding of statistical behavior across diverse scenarios.

Keywords: Key words and phrases. Tsallis statistics; q-logarithm; Pearson equation.

1. Introduction

The Karl Pearson differential equation, renowned for generating key distributions in classical statistics, plays a pivotal role in the framework of statistical theory. Recent advances in nonextensive statistics, as pioneered by Tsallis in 1988 [10], have expanded our comprehension of statistical mechanics in complex systems.

- **Motivation and Objectives:** This paper stands at the crossroads of historical statistical foundations and contemporary advancements in nonextensive statistics. Our primary objectives are twofold: firstly, to explore the assimilation of Pearson's differential equation within the framework of Tsallis statistics, probing the possibility of bridging classical and nonextensive statistical realms. Secondly, we introduce a new perspective by incorporating the q-logarithm into the Pearson equation, resulting in the unveiling of novel qdistributions that offer fresh insights into statistical modeling.
- **Addressing Fundamental Questions:** In this journey, we address fundamental questions: Does Pearson's equation encompass Tsallis statistics, encompassing familiar q-distributions such as qexponential, q-gamma, and q-normal distributions? Furthermore, can the integration of the q-logarithm within Pearson's equation lead to the emergence of new q-distributions, akin to the q-exponential type 2 (qexp2) distribution?

2. Tsallis statistics

The q-exponential and q-logarithm functions are defined as [9]

$$\exp_q(x) = \begin{cases} \exp(x), \quad q = 1, \\ [1+(1-q)x]^{\frac{1}{1-q}}, \quad q \neq 1, \ 1+(1-q)x \geq 0, \\ 0, \ q \neq 1, \ 1+(1-q)x < 0, \end{cases}$$

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and

$$\text{In}_q(x) = \begin{cases} \text{In}(x), & x > 0,\ q = 1, \\ \frac{x^{1-q} - 1}{1-q}, & q \neq 1, x > 0, \\ \text{undefined}, & x \leq 0. \end{cases}$$

The mathematics of nonextensive statistical mechanics are the q-operations:

$$x +_q y = x + y + (1 - q)xy,$$

$$x -_q y = \frac{x - y}{1 + (1-q)y},$$

$$x \times_q y = \max\left\{(x^{1-q} + y^{1-q} - 1)^{\frac{1}{1-q}}, 0\right\} := (x^{1-q} + y^{1-q} - 1)_+^{\frac{1}{1-q}},$$

$$x \div_q y = (x^{1-q} - y^{1-q} + 1)_+^{\frac{1}{1-q}}.$$

We know that the solution of the differential equation dy dx = y is the exponential function exp x. In [1, 11], the author proposed the following equation

$$\frac{dy}{dx} = y^q$$

whose solution leads to the q-exponential, $y = \exp_q(x)$. In [1] also introduced the operator for q-derivative as follows

(2.1) $$\frac{df(x)}{d_q(x)} = \lim_{y \to x} \frac{f(x) - f(y)}{x - y} = [1 + (1 - q)x]\frac{df(x)}{d(x)},$$

Where

$$x -_q y = x +_q (-_q y) = \frac{x - y}{1 + (1-q)y} \left(y \neq \frac{1}{q-1}\right).$$

3. Exploring Pearson's Equation and q-Distributions

In this section, we outline our motivation for exploring the extension of Pearson's differential equation within the realm of Tsallis statistics and the incorporation of the q-logarithm. We pose fundamental questions about the connection between Pearson's equation and well-known q-distributions, paving the way for our subsequent investigation.
At the end of the 19th century, Karl Pearson presented the classic Pearson differential equation. Most of the famous families of continuous probability distributions, such as normal distributions, beta distributions, and gamma distributions, are derived from the Pearson equation. For more details on the Pearson system of continuous probability distributions, the interested readers are referred to [2, 3, 7], and their references.
In paper [8], researchers derived a new family of distributions based on the generalized Pearson differential equation, which is a natural extension of the generalized inverse Gaussian distribution.
In this article, we examine the following Pearson's differential equation

(3.1) $$\frac{df(x)}{dx} = \frac{\alpha 0 - x}{\alpha 1 + \alpha 2x + \alpha 3x^2} f(x).$$

By proper choice of the parameters α0, α1, α2, and α3, the most of important distributions of statistics can be generated from the equation (3.1).
(a) the exponential distribution when α0 = α1 = α3 = 0 and α2 > 0;
(b) the gamma distribution when α1 = α3 = 0, α2 > 0, and α0 > −α2;
(c) the normal distribution when α0 = α2 = α3 = 0 and α1 > 0.

In this paper, we will expand Pearson's equation (3.1) by using non-extended q-distributions from the Tsalis statistics.
The first goal of this paper is to present the q-distributions of the Pearson equation such as q-exponential, q-gamma and q-normal distributions.
Now, for α0 = α1 = α3 = 0, and α2 = 1, we have

$$\frac{df(x)}{dx} = -f(x). \tag{3.2}$$

Clearly, the solution of (3.2) is the exponential distribution f(x) = exp (−x). If we use the q-derivative, i.e.,

$$\frac{df(x)}{d_q(x)} = -f(x),$$

then due to (2.1), we have

$$\frac{df(x)}{dx} = \frac{-f(x)}{(x - qx + 1)}.$$

So,

$$\ln f(x) = \ln(1 + (1-q)x)^{\frac{1}{q-1}} + C,$$

therefore, if x > 0 and C = ln q, we have

$$f(x) = q(1 + (1-q)x)^{\frac{-1}{1-q}} = \frac{q}{\exp_q(x)}, 0 < q < 1.$$

The above equation is the q-exponential distribution. If $\acute{q} = 2 - q$, then

$$f(x) = (2 - \acute{q})(1 + (\acute{q} - 1)x)^{\frac{1}{1-\acute{q}}} = (2 - \acute{q}) \exp_{\acute{q}}(-x), 1 < \acute{q} < 2,$$

which was introduced by Tsallis [9]
Now, look at (3.2) again. It is easy to see that

$$\frac{df(x)}{dx} = -f(x) \leftrightarrow \frac{d \ln f(x)}{dx} = -1.$$

Our second goal in this paper is to investigate the existence of new qdistributions. In other words, our goal is to use the following q-logarithm, i.e.,

$$\frac{d\, \mathrm{In}_q\, f(x)}{d(x)} = -1, \tag{3.3}$$

which leads to the extraction of new q-distributions. In the next section, we will prove our claims.

We formulate the following questions that guide our exploration:

- **First question:** Does Pearson's equation (3.1) yield the well-known q-distributions in Tsallis statistics, such as q-exponential, q-gamma, and q-normal distributions? We address this query in the subsequent section.
- **Second question:** For specific parameter values, can we extract new q-distributions by incorporating the q-logarithm into Pearson's equation (3.1)? The answer to this question unfolds in the forthcoming section.

4. Main Result

4.1. Answering the First Question.

The following propositions show that Pearson's equation (3.1) yield the well-known q-distributions of Tsallis statistics such as q-exponential, q-gamma and q-normal distribution.

Proposition 1. Assume that $\alpha_3 = q-1$, $\alpha_0 = \alpha_1 = 0$ and $\alpha_2 = b > 0, x \geq 0$, $q < 2$, then (3.1) reduces to

$$\frac{df(x)}{dx} = -\frac{1}{b + (q-1)x} f(x)$$

which yields the probability density function of q-exponential distribution

$$f(x) = \frac{2-q}{b} \exp_q\left(-\frac{1}{b} x\right).$$

In partcular, if $q = \frac{\alpha+2}{\alpha+1}, \alpha > 0$ and $b = \beta(q-1), \beta > 0, 1 < q < 2$, then

$$f(x) = \frac{\alpha}{\beta}\left(1 + \frac{\alpha}{\beta}\right)^{-\alpha-1},$$

which is called the probability density function of Lomax distribution.
Proof. Since

$$\frac{df(x)}{dx} = -\frac{1}{b - x + qx} f(x),$$

then

$$\frac{\mathbf{d}\ln \mathbf{f(x)}}{\mathbf{dx}} = -\frac{\mathbf{1}}{\mathbf{b + x(q-1)}}.$$

Therefore,

$$\ln f(x) = \frac{1}{1-q}\ln\left[\frac{1}{q-1}(b - x + qx)\right] + C$$

$$= \frac{1}{1-q}\ln\left[\frac{1}{q-1}(b - x + qx)\right] - \frac{\ln\frac{b}{q-1}}{1-q} + \frac{\ln\frac{b}{q-1}}{1-q} + C$$

$$= \frac{1}{1-q}\ln\left[\frac{1}{q-1}(b - x + qx)\right] + \acute{C}$$

$$= \ln \exp_q\left(-\frac{1}{b}x\right) + \acute{C}$$

therefore, for $\acute{C} = \ln\left(\frac{2-q}{b}\right)$, we have

$$f(x) = \exp_q\left(-\frac{1}{b}x\right)\cdot\exp(\acute{C}) = \frac{2-q}{b}\exp_q\left(-\frac{1}{b}x\right).$$

if $q = \frac{\alpha+2}{\alpha+1}$, $\alpha > 0$ and $b = \beta(q-1), \beta > 0, 1 < q < 2$, then

$$f(x) = \frac{2-q}{b}\exp_q\left(-\frac{1}{b}x\right)$$

$$= \frac{2-\frac{\alpha+2}{\alpha+1}}{\beta\left(\frac{\alpha+2}{\alpha+1}-1\right)}\exp_{\frac{\alpha+2}{\alpha+1}}\left(-\frac{x}{\beta(q-1)}\right)$$

$$= \frac{2-\frac{\alpha+2}{\alpha+1}}{\beta\left(\frac{\alpha+2}{\alpha+1}-1\right)}\left(1 - \left(1 - \frac{\alpha+2}{\alpha+1}\right)\frac{x}{\beta(q-1)}\right)^{\frac{1}{1-\frac{\alpha+2}{\alpha+1}}}$$

$$= \frac{\alpha}{\beta}\left(1 + \frac{\alpha}{\beta}\right)^{-\alpha-1},$$

which is the probability density function of Lomax distribution.

Proposition 2. Assume that $\alpha_0 = \frac{bd}{b+d-dq}$, $\alpha_1 = 0, \alpha_2 = \frac{b^2}{b+d-dq}$ and $\alpha_3 = \frac{(q-1)b}{b+d-dq}$, $b > 0, d > -b, x \geq 0$ then (3.1)reduces to

$$\frac{df(x)}{dx} = \frac{\frac{bd}{b+d-dq} - x}{\frac{b^2x}{b+d-dq} + \frac{(q-1)b}{b+d-dq}x^2} f(x)$$

which yields the probability density function of q-gamma distribution as follows

$$f(x) = kx^{\frac{d}{b}} \cdot \exp_q -\frac{x}{b},$$

where k is constant.

Proof. Since

$$\frac{df(x)}{dx} = \frac{\frac{bd}{b+d-dq} - x}{\frac{b^2 x}{b+d-dq} + \frac{(q-1)b}{b+d-dq} x^2} f(x),$$

Then

$$\frac{d\ln(f(x))}{dx} = \frac{d(b-x+qx) - bx}{bx(b-x+qx)}$$

Therefore

$$\ln(f(x)) = \frac{1}{1-q} \ln\left[\frac{1}{1-q}(b-x+qx)\right] + \frac{d}{b}\ln(x) + c$$

$$= \ln\left[\exp_q\left(-\frac{1}{b}\, x\right)\right] + \ln\left(x^{\frac{d}{b}}\right) + c$$

$$= \ln\left[x^{\left(\frac{d}{b}\right)} \exp_q\left(-\frac{1}{b}\, x\right)\right] + \acute{c}\,.$$

Therefor, for $\acute{c} = \ln\left[\frac{\left(\frac{1}{b}\right)^{\frac{d}{b}+1} \Gamma\left(\frac{1}{q-1}\right)(q-1)^{\frac{d}{b}+1}}{\Gamma\left(\frac{d}{b}+1\right)\Gamma\left(\frac{1}{q-1} - \left(\frac{d}{b}+1\right)\right)}\right]$, we have

$$f(x) = \exp(\acute{c})x^{\frac{d}{b}} \cdot \exp_q\left(-\frac{x}{b}\right) = kx^{\frac{d}{b}} \cdot \exp_q\left(-\frac{x}{b}\right),$$

where k is constant. This yields the q-gamma distribution

Proposition 3. If $\alpha_0 = 0, \alpha_1 = a, \alpha_2 = 0$ and $\alpha_3 = \frac{q-1}{2}$, $x \in (-\infty, \infty), a > 0, q < 3$ then(3.1) reduces to

$$\frac{df(x)}{dx} = \frac{-x}{\alpha + \frac{q-1}{2} x^2} f(x)$$

which yields the probability density function of q-normal distribution as follows

$$f(x) = \frac{1}{\sqrt{2a}c_q} \exp_q\left(-\frac{x^2}{2a}\right).$$

Proof. Since

$$\frac{df(x)}{dx} = \frac{-x}{\alpha + \frac{q-1}{2} x^2} f(x)$$

Then

$$\frac{d\ln(f(x))}{dx} = \frac{-2x}{2a + qx^2 - x^2}$$

Therefore

$$\ln(f(x)) = \frac{1}{1-q} \ln \frac{1}{1-q}(2a + qx^2 - x^2) + c$$

$$= \frac{1}{1-q} \ln\left(1 + (q-1)\frac{x^2}{2a}\right) + \acute{c}$$

$$= \ln\left(\exp_q\left(-\frac{x^2}{2a}\right)\right) + \acute{c},$$

therefore,for $\acute{c} = \ln\left(\frac{1}{\sqrt{2ac_q}}\right)$ we have

$$f(x) = \frac{1}{\sqrt{2ac_q}} \exp_q\left(-\frac{x^2}{2a}\right)$$

which yields the q-normal distribution.

4.2. Answering the Second Question.

Theorem 4. The equation (3.3) yields the probability density function of the following q-exponential distribution

$$f(x) = \exp_q\left(-x + \ln_q\left(2 - q^{\frac{1}{2-q}}\right)\right), 1 \le q < 2.$$

Proof. Since

$$\frac{d}{dx} \ln_q (f(x)) = -1,$$

Then

$$\ln_q\big(f(x)\big) = -x + C.$$

if $C = \frac{(2-q)^{\frac{1-q}{2-q}} - 1}{1-q} = \ln_q\left((2-q)^{\frac{1}{2-q}}\right)$, we have

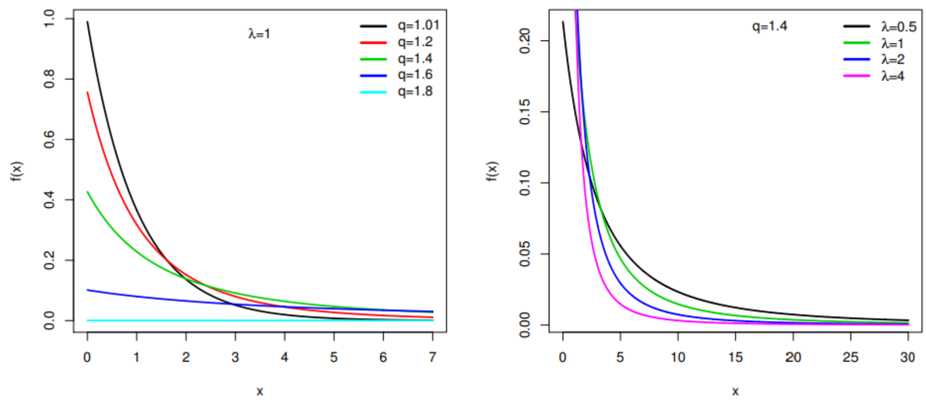
$$f(x) = \left[1 + (q-1)\left(x - \frac{2 - q^{\frac{1-q}{2-q}} - 1}{1-q}\right)\right]^{\frac{1}{1-q}}$$

$$= \exp_q\left(-x + \ln_q\left(2 - q^{\frac{1}{2-q}}\right)\right).$$

Definition 5. A random variable X is said to have the q-exponential distribution (type 2) or expq2 with parameter q, λ, denoted by X ~ qE (q, λ), if its pdf is given by

$$f_{qE}(q,\lambda) = \lambda \exp_q\left(-\lambda x + \ln_q\left((2-q)^{\frac{1}{2-q}}\right)\right), x > 0, \lambda > 0, 1 \le q \le 2.$$

The graphs of the probability density function of q-exponential distribution (type2) are presented for different values of parameters q, λ and shown in figure1.



(A) for fixed λ and different q (B) for fixed q and different λ

Figure 1. Graphs of Probability density function for qexponential distribution(type2) with parameters q, λ.

5. Simulation and Comparative Analysis

To assess the effectiveness of the qexp2 distribution, we employ the MetropolisHastings algorithm to generate a random sample of size n = 100. The generated data follow the qexp2 distribution with parameters q = 1.4 and λ = 1. This simulated dataset is presented in Table 1.
Furthermore, we evaluate the performance of various distributions in fitting the simulated data. Specifically, we consider the Weibull, Gamma, Lomax, qexp, and qexp2 distributions. To guide our comparison, we utilize the Akaike Information Criterion (AIC) as a measure of goodness-of-fit. The calculated AIC values for each distribution are presented in Table 2.
Notably, the results reveal intriguing insights. The AIC values for the qexp and qexp2 distributions are remarkably similar, differing only in the fifth decimal place. This intriguing similarity underscores the effectiveness of the qexp2 distribution in modeling complex phenomena.

Table 1. Simulated data (n = 100) following the qexp2 distribution with parameters q = 1.4 and λ = 1

0.592	3.047	3.407	5.938	2.484	4.01	2.146	0.36	0.94	2.639
4.335	4.195	2.179	2.763	1.554	0.784	0.707	1.644	4.41	2.15
7.904	8.211	8.482	10.283	6.412	6.037	3.98	5.659	3.047	3.939
0.397	0.946	0.335	1.658	1.34	3.834	5.503	5.399	9.611	10.952
11.807	12.469	14.42	12.225	10.391	7.973	5.071	9.322	11.677	14.805
6.835	0.909	1.357	2.322	2.99	6.645	7.466	8.539	8.95	7.091
0.882	1.351	3.007	4.57	5.77	1.12	0.011	4.066	6.452	8.099
2.468	0.527	0.119	2.297	0.954	1.664	1.498	0.345	2.168	3.433
10.928	9.791	6.705	7.167	6.573	6.3	3.705	4.046	5.95	7.089
15.418	13.826	18.287	18.212	18.126	15.39	14.133	14.4	10.669	8.243

Table 2. The AIC and AICc values for all fitted distributions, considering the simulated data

	exp_q	$exp_q 2$	Lomax	Gamma	weibull
AIC	548.47898	548.47895	556.69	554.67	553.33
AICc	548.60269	548.60266	556.81	554.79	553.46

6. Application to Real-world Data

In this section, we present a comprehensive application of the qexp2 distribution using the Akaike Information Criterion (AIC) to assess its performance in real-world scenarios. We analyze two distinct datasets, employing rigorous statistical methodologies to determine the distribution that best characterizes the underlying phenomena.

Dataset 1: Air Conditioning System Failure Times

We begin by examining a dataset derived from Linhart and Zucchini (1986), which records failure times of an aircraft's air conditioning system. This dataset, presented in Table 3, serves as an illustrative case for our analysis.

Dataset 2: WTI Crude Oil Price Differences

Our investigation extends to a second dataset obtained from Mehri-Dehnavi, Agahi, and Mesiar (2019). Spanning from January 2, 1986, to July 3, 2017, this dataset comprises absolute differences between the WTI crude oil price and its last 100 days' moving averages. With a total of n = 7849 data points, it provides a real-world scenario to rigorously evaluate distribution fitting.

Model Evaluation and AIC Comparison

We evaluate the suitability of five distributions: the qexp, qexp2, Lomax, gamma, and Weibull distributions. Employing the AIC as a robust criterion, we compare the goodness-of-fit for each distribution on both datasets.

Results and Insights

The results of our analysis are presented in Table 4 for Dataset 1 and Table 5 for Dataset 2. Notably, our findings unveil compelling insights. The qexp2 distribution demonstrates noteworthy similarities with the qexp and Lomax distributions, hinting at its potential in characterizing diverse phenomena.
In summary, our comprehensive application of the qexp2 distribution, guided by rigorous statistical analysis and the AIC, underscores its efficacy in modeling complex real-world datasets. This section serves as a testament to the distribution's versatility and potential across a range of applications.

Table 3. Dataset 1 - Failure times of an aircraft's air conditioning system

23	261	87	7	120	14	62	47	225	71	246	21	42	20	5
12	120	11	3	14	71	11	14	11	16	90	1	16	52	95

Table 4. AIC and AICc values for all fitted distributions on Dataset 1

	$\exp_q$	$\exp_q 2$	lomax	gamma	weibull
AIC	307.6749	307.6749	307.6749	308.3347	307.8738
AICc	308.1193	308.1193	308.1193	308.7791	308.3182

Table 5. AIC and AICc values for all fitted distributions on Dataset 2

	$\exp_q$	$\exp_q 2$	lomax	gamma	weibull
AIC	37412.7242582	37412.7242581	37412.7242585	37878.039	37716.971
AICc	37412.7257876	37412.7257875	37412.7257879	37878.04	37716.972

7. Conclusion

In this work, we have embarked on a journey that seamlessly bridges classical statistical theory with the intricate landscape of nonextensive statistics. Our investigation has unearthed a compelling link between the Pearson differential equation and pivotal distributions, highlighting its role in generating fundamental q-distributions within the framework of Tsallis statistics. Building upon this foundation, we have introduced a novel qdistribution, the q-exponential type 2 (qexp2), by extending the Pearson equation through the incorporation of the q-logarithm.
The synergy between theoretical exploration and practical application forms the cornerstone of our study. Our meticulous analysis, spanning bothsimulated and real-world datasets, unequivocally demonstrates the applicability and potential advantages of the proposed q-distributions. The qexp2 distribution, in particular, emerges as a promising tool for modeling diverse phenomena, showcasing its efficacy alongside established distributions.
This study ignites the spark of curiosity for future inquiries. The question of employing q-differentiation in conjunction with the q-logarithm to attain a more comprehensive q-Carl Pearson formula beckons further investigation. Additionally, the prospect of unearthing novel q-distributions remains tantalizing, promising to deepen our understanding of statistical modeling in complex systems.
In conclusion, our journey has led to the harmonious fusion of theory and application, unveiling new dimensions in statistical theory. As we lay the groundwork for future explorations, we hope this contribution not only enriches the mathematical landscape but also inspires meaningful strides in the realm of statistical sciences.

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