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Research Article

On The Crosscap Of Generalized Zero-Divisor Graph Of Commutative Rings

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ARTICLE INFO	ABSTRACT
	Let R be a commutative ring and $Z(R)^*$ be the set of all nonzero zero-divisors. The generalized zero divisor graph of R is defined as the graph $\Gamma_g(R)$ with vertex set $Z(R)^*$ and two vertices x and y are adjacent if and only if $ann_R(x) + ann_R(y)$ is an essential ideal of R. In this paper, we classify all finite commutative rings R for which $\Gamma_g(R)$ has crosscap one.

Introduction

Let R be a finite commutative ring. A nonempty subset I of R is said to be an *ideal* if $(I, +)$ is a subgroup of R and for every $a \in I$ and $r \in R$, $ra \in I$. A non-zero ideal I of R is called *essential*, denoted by $I \leq_e R$, if I has a non-zero intersection with any non-zero ideal of R.
By a graph $G = (V, E)$, we mean an undirected simple graph with vertex set V and edge set E. A graph in which each pair of distinct vertices is joined by the edge is called a complete graph. We use K_n to denote the complete graph with n vertices. An r–partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A complete r–partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2–partite graph) with part sizes m and n is denoted by $K_{m,n}$. If $G = K_{1,n}$, where $n \geq 1$, then G is a star graph.
By a surface, we mean a connected two-dimensional real manifold, i.e., a connected topological space such that each point has a neighbourhood homeomorphic to an open disk. It is well known that any compact surface is either homeomorphic to sphere, or to a connected sum of g tori, or to a connected sum of k projective planes. The orientable genus of a graph G, $g(G)$, is the minimum genus of a surface in which G can be embedded. The non-orientable genus of a graph G, $g(G)$, is the minimum non-orientable genus of a surface (crosscaps) in which G can be embedded. The number g is called the genus of the surface S_g and k is called the crosscap of N_k. A planar graph is a graph of genus (crosscap) 0, a toroidal graph is a graph of genus 1, and a projective graph is a graph of crosscap 1.

Literature review

In the literature, there are many papers assigning graphs to rings, groups and semi- groups, see [1, 2, 8, 12]. First graph construction from a commutative ring is the zero-divisor graph by Beck [8] and later studied by Anderson et al[2]. There are several other graphs associated with commutative rings and some of them to men- tion are total graph[3], annihilating-ideal graph[9] and annihilator graph [5]. Several authors [12, 14, 15, 16, 17, 19, 20] studied about various properties of these graphs including diameter, girth, domination and genus. In [6], Basharlou et al introduced the *generalized zero-divisor graph* $\Gamma_g(R)$. One can see that the zero-divisor graph $\Gamma(R)$ is a subgraph of the generalized zero-divisor graph $\Gamma_g(R)$. Some basic prop- erties like girth, diameter were discussed. All the rings with the same generalized zero-divisor and zero-divisor graph were characterized. Also, they proved that the generalized zero-divisor graph associated with an Artinian ring is weakly perfect. Further, K. Selvakumar et al [7] classified all finite commutative rings R for which $\Gamma_g(R)$ is planar. Also, all finite commutative rings R for which $\Gamma_g(R)$ has genus one were classified. In this paper, we characterized all finite commutative rings R whose crosscap is one.

Preliminaries

The following are useful in the sequel of this paper and hence given below.
Definition 3.1. *Let R be a commutative ring and $Z(R)^*$ be the set of all nonzero zero-divisors. The generalized zero divisor graph of R is defined as the graph $\Gamma_g(R)$ with vertex set $Z(R)^*$ and two vertices x and y are adjacent if and only if $ann_R(x) + ann_R(y)$ is an essential ideal of R.*

Theorem 3.2. If $m \geq 2$ and $n \geq 2$, then

$$g(K_{m,n}) = \lceil \frac{(m-2)(n-2)}{4} \rceil.$$

If $m \geq 3, n \geq 3$, then

$$\bar{g}(K_{m,n}) = \lceil \frac{(m-2)(n-2)}{2} \rceil.$$

Lemma 3.3. Let G be a connected graph with $n \geq 3$ vertices and q edges. Then

$$\gamma(G) \geq \lceil \frac{q}{2} - \frac{n}{2} + 1 \rceil \quad \text{and} \quad \bar{\gamma}(G) \geq \lceil \frac{q}{3} - n + 2 \rceil.$$

If G contains no cycle of length 3, then

$$\gamma(G) \geq \lceil \frac{q}{4} - \frac{n}{2} + 1 \rceil \quad \text{and} \quad \bar{\gamma}(G) \geq \lceil \frac{q}{2} - n + 2 \rceil.$$

Lemma 3.4. let R be a finite ring which is not local; then $\gamma(\Gamma(R)) = 1$ if and only if R is isomorphic to one of the following 29 types of rings:
$\mathbb{F}_4 \times \mathbb{F}_4, \mathbb{F}_4 \times \mathbb{Z}_5, \mathbb{F}_4 \times \mathbb{Z}_7, \mathbb{Z}_5 \times \mathbb{Z}_5, \mathbb{Z}_2 \times \frac{\mathbb{F}_4[x]}{<x^2>}, \mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{<x^2+x+1>}, \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x,y]}{<x^2,xy,y^2>}, \mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{<2x,x^2>}, \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{<x^3>}, \mathbb{Z}_3 \times \frac{\mathbb{Z}_4[x]}{<x^2-2,x^3>}, \mathbb{Z}_3 \times \mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{F}_4, \mathbb{F}_4 \times \frac{\mathbb{Z}_2[x]}{<x^2>}, \mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_4 \times \frac{\mathbb{Z}_2[x]}{x^2}, \frac{\mathbb{Z}_2[x]}{x^2} \times \frac{\mathbb{Z}_2[x]}{x^2}, \mathbb{Z}_4 \times \mathbb{Z}_5, \mathbb{Z}_5 \times \frac{\mathbb{Z}_2[x]}{<x^2>}, \mathbb{Z}_4 \times \mathbb{Z}_7, \mathbb{Z}_7 \times \frac{\mathbb{Z}_2[x]}{<x^2>}, \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{F}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

4 Crosscap of $\Gamma_g(R)$

In this section, we classify all finite commutative rings whose $\Gamma_g(R)$ has crosscap is 1.

Theorem 4.1. Let $R = F_1 \times F_2 \times \ldots \times F_n$ be a finite commutative ring with identity, where each F_i, where $1 \leq i \leq n$, is a field. Then $\bar{g}(\Gamma_g(R)) = 1$ if and only if R is isomorphic to one of the following rings:
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4, \mathbb{F}_4 \times \mathbb{F}_4, \mathbb{F}_4 \times \mathbb{Z}_5$.

Proof. Let $R = F_1 \times F_2 \times \ldots \times F_n$, where each F_i, where $1 \leq i \leq n$, is a field. Let $n \geq 5$. Suppose $n = 5$. Let $|R_i| = 2 \forall i = 1, 2, \ldots, 5$. Then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Let $a_1 = (1,0,0,0,0), a_2 = (0,1,0,0,0), a_3 = (1,1,0,0,0), b_1 = (0,0,1,0,0), b_2 = (0,0,0,1,0), b_3 = (0,0,0,0,1), b_4 = (0,0,1,1,0), b_5 = (0,0,0,1,1), b_6 = (0,0,1,0,1), b_7 = (0,0,1,1,1)$. Here $a_i b_j = 0$ for $i = 1,2,3$ and $j = 1,2,\ldots,7$ so that $K_{3,7} \subset \Gamma_g(R)$ which implies that $\bar{g}(\Gamma_g(R)) \geq 2$. Therefore, $n \leq 4$.

Consider $n = 4$. Then by [7, Theorem 3.1], $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is the only choice.

Since $\bar{g}(\Gamma_g(R)) \geq g(\Gamma_g(R))$, we have $\bar{g}(\Gamma_g(R)) \geq 1$. Let $u_1 = (1,0,0,0), u_2 = (0,1,0,0), u_3 = (1,1,0,0), v_1 = (0,0,1,0), v_2 = (0,0,0,1), v_3 = (0,0,1,1), w_1 = (1,0,1,0), w_2 = (1,0,0,1), w_3 = (0,1,1,0), w_4 = (0,1,0,1)$. Here $u_iv_j = 0$, where $i = 1,2,3; j = 1,2,3$, so that $K_{3,3} \subset \Gamma_g(R) \geq 1$. Here, $E(G) = \{u_iv_j : i,j = 1,2,3\} \cup \{w_1u_2, w_1v_2, w_2u_2, w_2v_1, w_3v_2, w_3u_1, w_4u_1, w_4v_1\} \cup \{w_1w_4, w_2w_3\}$. Now, the embedding of $\Gamma_g(R)$ in the projective plane is given in Figure 1. Therefore, $\bar{g}(\Gamma_g(R)) = 1$.

Consider $n = 3$. Then by [7, Theorem 3.1], $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7$ are the only choice.

For $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7$, let $a_1 = (1,0,0), a_2 = (0,1,0), a_3 = (1,1,0), b_1 = (0,0,1), b_2 = (0,0,2), b_3 = (0,0,3), b_4 = (0,0,4), b_5 = (0,0,5), b_6 = (0,0,6)$. Here $a_ib_j = 0$, where $i = 1,2,3; j = 1,2,\ldots,6$, so that $K_{3,6} \subset \Gamma_g(R)$. Therefore, $\bar{g}(\Gamma_g(R)) \geq 2$.

For $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, let $G = \Gamma_g(R)$. Let $u_1 = (1,0,0), u_2 = (0,0,1), u_3 = (0,2,0), v_1 = (0,1,0), v_2 = (2,0,0), v_3 = (0,0,2), W_1 = \{(0,1,1),(0,1,2),$

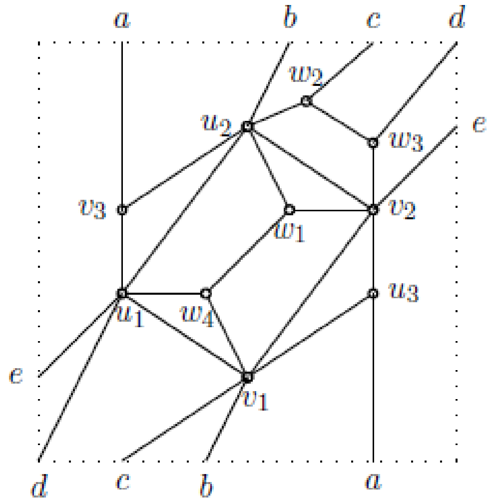

Figure 1: $\Gamma_g(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ *in* N_1

$(0,2,1),(0,2,2)\}, W_2 = \{(1,0,1),(1,0,2),(2,0,1),(2,0,2)\}$ and $W_3 = \{(1,1,0), (1,2,0),(2,1,0),(2,2,0)\}$. Since $g(\Gamma_g(R)) = 1$, we have $\bar{g}(\Gamma_g(R)) \geq 1$. Now, the embedding in the projective plane is given in Figure 2. Therefore, $\bar{g}(\Gamma_g(R)) = 1$.

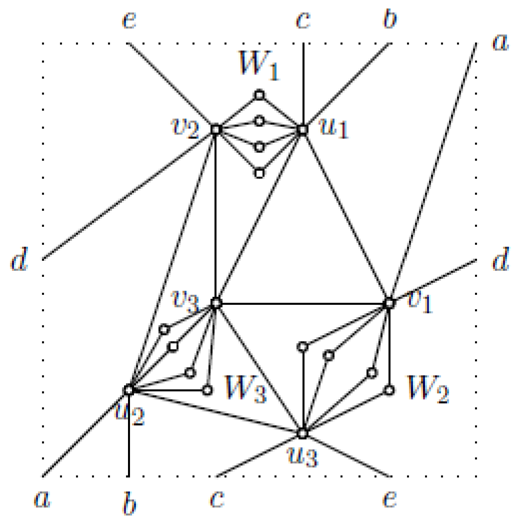

Figure 2: $\Gamma_g(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$ *in* N_1

For $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, let $G = \Gamma_g(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$. Since $g(\Gamma(R)) = 1$, we have $\bar{g}(\Gamma_g(R)) \geq 1$. Moreover, G is the subgraph of $\Gamma_g(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$. Therefore, we have $\bar{g}(\Gamma_g(R)) = 1$.

For $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$, let $G = \Gamma_g(R)$. Now, let $u_1 = (1,0,0), u_2 = (0,1,0), u_3 = (1,1,0), v_1 = (0,0,1), v_2 = (0,0,2), v_3 = (0,0,3), v_4 = (0,0,4), w_1 = (1,0,1), w_2 = (1,0,2), w_3 = (1,0,3), w_4 = (1,0,4), w_5 = (0,1,1), w_6 = (0,1,2), w_7 = (0,1,3), w_8 = (0,1,4)$. Here $u_i v_j = 0$, where $i = 1,2,3; j = 1,\ldots,4$, so that $K_{3,4} \subseteq \Gamma_g(R)$. Therefore, $\bar{g}(\Gamma_g(R)) \geq 1$. Also, $\bar{G} = G - \{w_1, w_2, \ldots, w_8\}$ can be embedded in the projective plane in Figure 1. Therefore, $\bar{g}(\Gamma_g(R)) = 1$.

For $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$, let $G = \Gamma_g(R)$. Now, let $u_1 = (1,0,x_0), u_2 = (0,1,x_0), u_3 =$

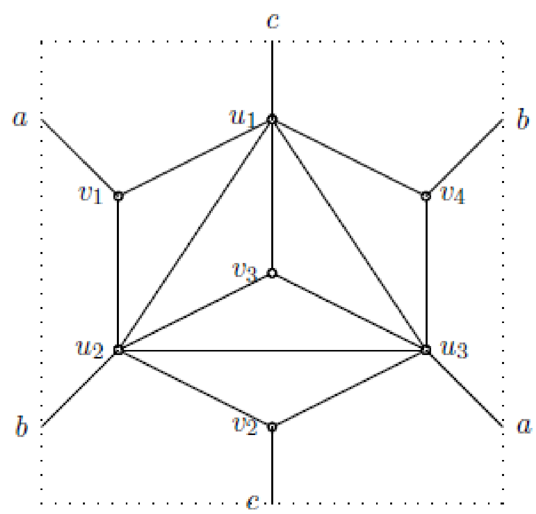

Figure 3: $\Gamma_g(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5)$ *in* N_1

$(1,1,x_0), v_1 = (0,0,x_1), v_2 = (0,0,x_2), v_3 = (0,0,x_3)$, $w_1 = (1,0,x_1), w_2 = (1,0,x_2), w_3 = (1,0,x_3), w_4 = (0,1,x_1), w_5 = (0,1,x_2)$, $w_6 = (0,1,x_3)$. Here $u_iv_j = 0$, where $i = 1,2,3; j = 1,\ldots,3$, so that $K_{3,3} \subset \Gamma_g(R)$. Therefore, $\bar{g}(\Gamma_g(R)) \geq 1$. Also, $\Gamma_g(R)$ is the subgraph of $\Gamma_g(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4)$. Therefore, $\bar{g}(\Gamma_g(R)) = 1$.

Consider $n = 2$. Then, from Lemma 3.4, $\mathbb{F}_4 \times \mathbb{F}_4, \mathbb{F}_4 \times \mathbb{Z}_5, \mathbb{F}_4 \times \mathbb{Z}_7, \mathbb{Z}_5 \times \mathbb{Z}_5$ are the only choice.

For $R \cong \mathbb{F}_4 \times \mathbb{Z}_7$, let $G = \Gamma_g(R)$. Let $u_1 = (x_1,0), u_2 = (x_2,0), u_3 = (x_3,0), v_1 = (x_0,1), v_2 = (x_0,2), v_3 = (x_0,3), v_4 = (x_0,4), v_5 = (x_0,5)$, $v_6 = (x_0,6)$. Here, $u_iv_j = 0$ for every $i = 1,2,3; j = 1,\ldots,6$, so that $K_{3,6} \subseteq \Gamma_g(R)$. Therefore, by Lemma 3.2, we get $\bar{g}(G) \geq 2$.

For $R \cong \mathbb{F}_4 \times \mathbb{Z}_5$, let $G = \Gamma_g(R)$. Let $u_1 = (x_1,0), u_2 = (x_2,0), u_3 = (x_3,0), v_1 = (x_0,1), v_2 = (x_0,2), v_3 = (x_0,3), v_4 = (x_0,4)$. Here, $u_iv_j = 0$ for every $i = 1,2,3; j = 1,\ldots,4$, so that $K_{3,4} \subseteq \Gamma_g(R)$. Therefore, $\bar{g}(G) \geq 1$. Also, the embedding of $\Gamma_g(R)$ in the projective plane is given in Figure 3.

For $R \cong \mathbb{F}_4 \times \mathbb{F}_4$, let $G = \Gamma_g(R)$. Let $u_1 = (x_1,x_0), u_2 = (x_2,x_0), u_3 = (x_3,x_0), v_1 = (x_0,x_1), v_2 = (x_0,x_1), v_3 = (x_0,x_3)$. Here, $u_iv_j = 0$ for every $i = 1,2,3; j = 1,\ldots,3$, so that $K_{3,3} \subseteq \Gamma_g(R)$. Therefore, $g(G) \geq 1$. Also, we see that $\Gamma_g(\mathbb{F}_4 \times \mathbb{F}_4) \subset \Gamma_g(\mathbb{F}_4 \times \mathbb{Z}_5)$. Therefore, we get $\bar{g}(\Gamma_g(R)) = 1$.

For $R \cong \mathbb{Z}_5 \times \mathbb{Z}_5$, let $G = \Gamma_g(R)$. Let $u_1 = (1,0), u_2 = (2,0), u_3 = (3,0), u_4 = (4,0)v_1 = (0,1), v_2 = (0,2), v_3 = (0,3), v_4 = (0,4)$. Here, $u_iv_j = 0$ for every $i = 1,\ldots,4; j = 1,\ldots,4$, so that $K_{4,4} \subseteq \Gamma_g(R)$. Therefore, Therefore, by Lemma 3.2, we get $\bar{g}(G) \geq 2$.

□

Theorem 4.2. Let $R = R_1 \times R_2 \times \ldots \times R_n$ be a finite commutative ring with identity, where each R_i, where $1 \leq i \leq n$, is a finite local rings. Then $\bar{g}(\Gamma_g(R)) = 1$

if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{<2x,x^2>}, \mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x,y]}{<x^2,xy,y^2>}, \mathbb{Z}_2 \times \frac{\mathbb{F}_4[x]}{<x^2>}, \mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{<x^2+x+1>}, \mathbb{Z}_4 \times \mathbb{F}_4, \frac{\mathbb{Z}_2[x]}{<x^2>} \times \mathbb{F}_4$.

Proof. For $n \geq 4$, there is no rings in this case by [7, Theorem 2.1].

Suppose $n = 3$. Then $R \cong R_1 \times R_2 \times R_3$, where (R_i, m_i) is a local ring for $i = 1,2,3$. Then by [7, Theorem 2.1], $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{x^2}$ is the only case.

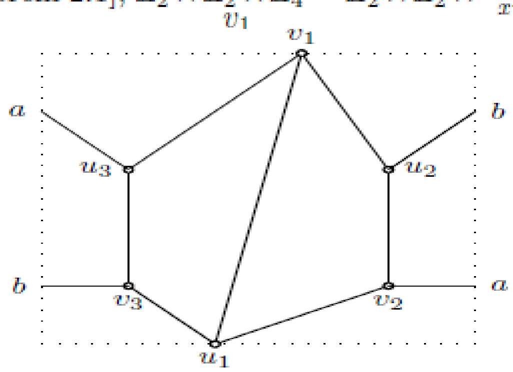

Figure 4: $\Gamma_g(\mathbb{Z}_4 \times \mathbb{F}_4)$ *in* N_1

For $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$, let $u_1 = (1,0,0), u_2 = (0,1,0), u_3 = (1,1,0), u_4 = (0,1,2), u_5 = (1,0,2), v_1 = (0,0,1), v_2 = (0,0,2), v_3 = (0,0,3)$. Here $u_iv_j = 0$ for every $i = 1,2,\ldots 5, j = 1,2,3$, so that $K_{3,5} \subseteq \Gamma_g(R)$ and so $\bar{g}(\Gamma_g(R)) \geq 2$ by Theorem 3.2.

Suppose $n = 2$. From [7, Theorem 2.1], $\mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_4 \times \frac{\mathbb{Z}_2[x]}{<x^2>}, \frac{\mathbb{Z}_2[x]}{<x^2>} \times \frac{\mathbb{Z}_2[x]}{<x^2>}, \mathbb{Z}_3 \times \mathbb{Z}_8, \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{x^3}, \mathbb{Z}_3 \times \frac{\mathbb{Z}_4[x]}{<x^2-2,x^3>}, \mathbb{Z}_4 \times \mathbb{F}_4, \frac{\mathbb{Z}_2[x]}{<x^2>} \times \mathbb{F}_4, \mathbb{Z}_2 \times \frac{\mathbb{F}_4[x]}{<x^2>}, \mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{<x^2+x+1>}, \mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x,y]}{<x^2,xy,y^2>}, \mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{<x^2,2x>}$ are the only choice.

For $R \cong \mathbb{Z}_4 \times \mathbb{Z}_4$, let $u_1 = (1,0), u_2 = (2,0), u_3 = (3,0), u_4 = (2,2), v_1 = (0,1), v_2 = (0,2), v_3 = (0,3), x_1 = (2,1), x_2 = (2,3), x_3 = (1,2), x_4 = (3,2)$. Here $u_iv_j = 0$ for $i = 1,2,3,4; j = 1,2,3$ so that $K_{3,4} \subseteq \Gamma_g(R)$. Here, $E(G) = \{u_iv_j | i = 1,\ldots,4; j = 1,2,3\} \cup \{u_1u_2, u_2u_3, u_1u_4, u_2u_4, u_3u_4\} \cup \{x_1u_2, x_1u_4, x_1v_2, x_2u_1, x_2u_2, x_2u_4, x_3u_4, x_3v_2, x_4u_4, x$
$\{x_1x_4, x_2x_3, x_2x_4\}$. Also, $|E(G)| = 38$ and $|V(G)| = 11$. Therefore, by Lemma 3.3, we get $\bar{g}(\Gamma_g(R)) \geq 3$.

Since $\Gamma_g(\mathbb{Z}_4 \times \mathbb{Z}_4) \cong \Gamma_g(\mathbb{Z}_4 \times \frac{\mathbb{Z}_2[x]}{<x^2>}) \cong \Gamma_g(\frac{\mathbb{Z}_2[x]}{<x^2>} \times \frac{\mathbb{Z}_2[x]}{<x^2>})$, we get $\bar{g}(\Gamma_g(\mathbb{Z}_4 \times \frac{\mathbb{Z}_2[x]}{<x^2>})) = \bar{g}(\Gamma_g(\frac{\mathbb{Z}_2[x]}{<x^2>} \times \frac{\mathbb{Z}_2[x]}{<x^2>})) \geq 2$.

For $R \cong \mathbb{Z}_4 \times \mathbb{F}_4$, let $G = \Gamma_g(R)$. Now, let $a_1 = (1, x_0), a_2 = (2, x_0), a_3 = (3, x_0), b_1 = (0, x_1), b_2 = (0, x_2), b_3 = (0, x_3), c_1 = (2, x_1), c_2 = (2, x_2), c_3 = (2, x_3)$. Here $a_ib_j = 0$ for every $i = 1,2,3; j = 1,2,3$, so that $K_{3,3} \subseteq \Gamma_g(R)$ and so $\bar{g}(\Gamma_g(R)) \geq 1$. However, $\tilde{G} = G - \{c_1, c_2, c_3\}$ can be embedded in the projective plane given in Figure 4. Therefore, $\bar{g}(\Gamma_g(R)) = 1$.

Since $\Gamma_g(\mathbb{Z}_4 \times \mathbb{F}_4) \cong \Gamma_g(\frac{\mathbb{Z}_2[x]}{<x^2>} \times \mathbb{F}_4)$, so that $\bar{g}(\Gamma_g(\frac{\mathbb{Z}_2[x]}{<x^2>} \times \mathbb{F}_4)) = 1$.

For $R \cong \mathbb{Z}_2 \times \frac{\mathbb{F}_4[x]}{<x^2>}$, let $G = \Gamma_g(R)$. Let $u_1 = (0, x), u_2 = (0, ax), u_3 = (0, bx), v_1 = (1, 0), v_2 = (1, x), v_3 = (1, ax), v_4 = (1, bx), w_1 = (0, 1), w_2 = (0, 1 + x), w_3 = (0, 1 +$

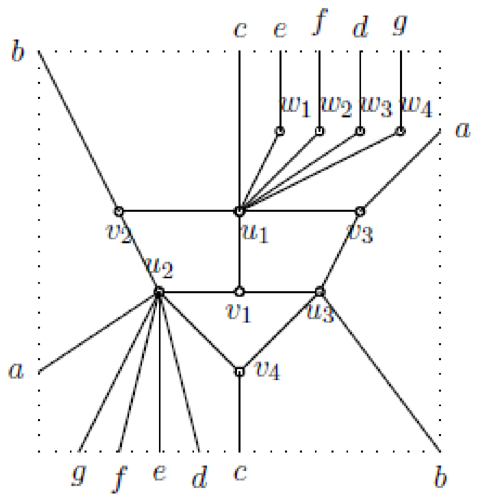

Figure 5: $\Gamma_g(\mathbb{Z}_3 \times \mathbb{Z}_8)$ *in* N_1

$ax), w_4 = (0, 1 + bx)$. Here $u_iv_j = 0$ for every $i = 1,2,3; j = 1,\ldots,4$, so that $K_{3,4} \subseteq \Gamma_g(R)$. Therefore, $\bar{g}(\Gamma_g(R)) \geq 1$. Also, $\tilde{G} = G - \{w_1, \ldots, w_4\}$ can be embedded in the projective plane in Figure 6. Therefore, $\bar{g}(G) = 1$. Since $\Gamma_g(\mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{<x^2+x+1>}) \cong \Gamma_g(R)$, we have $\bar{g}(\Gamma_g(\mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{<x^2+x+1>})) = 1$.

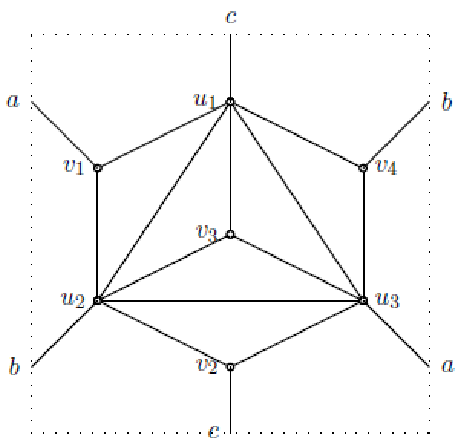

Figure 6: $\Gamma_g(\mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{<x^2,2x>}) \cong \Gamma_g(\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x,y]}{<x^2,xy,y^2>}) \cong \Gamma_g(\mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{<x^2+x+1>})$ *in* N_1

For $R \cong \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x,y]}{<x^2,xy,y^2>}$, let $G = \Gamma_g(R)$. Now, let $u_1 = (0,x), u_2 = (0,y), u_3 = (0,x+y), v_1 = (1,0), v_2 = (1,x), v_3 = (1,y), v_4 = (1,x+y), w_1 = (0,1), w_2 = (0,1+x), w_3 = (0,1+y), w_4 = (0,1+x+y)$. Here, $u_i.v_j = 0$ for $i = 1,2,3; j = 1,2,3,4$ so that $K_{3,4} \subset \Gamma_g(R)$. Therefore, $\bar{g}(\Gamma_g(R)) \geq 1$. However, we can draw $\bar{G} = g - \{w_1, \ldots, w_4\}$ in the projective plane given in Figure 6. Therefore, $\bar{g}(G) = 1$.

Since $\Gamma_g(\mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{<x^2,2x>}) \cong \Gamma_g(R)$, we have $\bar{g}(\Gamma_g(\mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{<x^2,2x>})) = 1$. □

Conclusion

In this article, we have studied generalized zero-divisor graphs $\Gamma_g(R)$ of finite com- mutative ring R. We classified all finite commutative rings whose $\Gamma_g(R)$ has crosscap one.

Data Availability Statement

The authors have not used any data for the preparation of this manuscript.

Declaration

Conflict of interest The authors declare that they have no conflicts of interest.

Competing Interests

The authors have no relevant financial or non-financial interest to disclose.

Authors Contribution

Both the authors contributed to design and implementation of this work. S. N. Meera drafted the analysis of the results and the first draft of the manuscript. Dr.
K. Selvakumar directed the interpretation of the results.

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