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Research Article

To Foster Mathematical Proficiency: Innovating Proof Strategies Within Mathematics Education Discourse

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ARTICLE INFO	ABSTRACT

We examine and combine etymological verification to show information in a longitudinal retrospective analysis. The data include 1,000 concepts, compared across 150 students. The problems are from the Academic Information System. The participants are learners in the pre-service graduate program for teachers covering the period from 2018 to 2022. Two research questions surface from the experimental review and word trajectories. Question 1 focused on identifying and creating proof models, while Question 2 explored word trajectories in contextual measures. The data are gathered from 400 proofs, divided into etymological sense and their understanding in exploring proof. The construct validity is assessed using the Test of Cramer, leading to a C coefficient of 0.83. The Plot Diagram examined interaction effects between variables through the ANOVA test. An intersection is identified based on the process of doing proof, with $\rho < \alpha = 0.05$, confirming model validity. The multiple determination value for all independent and dependent variables is 1, indicating a strong correlation. The results propose that the pattern leans towards a meta-pattern, with a non-linear description or variation in proof, and logical steps representing a form of thinking. In general, an increase in doing proof correlates with a decrease in mathematical representations.

Keywords: Etymological sense; learning experience; teaching and learning styles; language context;

1. Introduction

The core of proof is in the effective use of words and the establishment of logical connections with mathematical statements (Larkin and Simon, 1987). This includes understanding the meaning associated with proof and considering etymological sense (Rif'at, 2018; Rif'at et al., 2022). Approaches rooted in etymology pursue to explore connections across various language contexts, found in diverse teaching and learning styles such as textbooks and curriculum (Cooney and Wilson, 2020). However, the wide use of the term proof often confuses students requiring mastery as an ability (Puig and Gutierrez, 1996; Morton, 2018; Anat, Einav and Shirley, 2020; Piñeiro and Calle, 2023).

This research centers on adaptive proof, exploring the factors by observing words in action to show mathematical ideas. This includes informal explanations, justifications, and spontaneous, inductive reasoning based on words equivalent to proof. In addition, distinguishing significant results in logic from those in the etymological sense is challenging (Rif'at and Sugiatno, 2022). The outcome shows the difficulty of proof to some degree due to uncertainty. The inspector, who has 40 years of teaching experience, consistently encounter difficulties in providing proof and this sometimes leads to illogical arrangements of proof. However, one contributing factor are the influence of teaching culture, often relying on symbolically colored proof (Matitaputty, 2020; Beites, Branco and Costa, 2022). For instance, using attributes that often derived from assumption or defined-terms or theorem, usually or easier when presenting in symbols.

2. Literature Review

The etymological sense is to discover the connection between words or phrases and mathematical symbols or notations (Rif'at et al., 2022). In this case is in proving mathematics problems. That defines proof as the ability to find analogical correspondences of words in reasoning (Kaput, 2018). Analogy enables humans to gain understanding of unknown structures (proofs) by using knowledge of previously known structures (sources). Evidence supporting this point of view, together with the observation that analogy is pervasive in language and thought, suggests a key role for analogical processes at the core of human cognition (Indurkhya, 1992; Holyoak and Thagard, 1994; Gentner, Holyoak and Kokinov, 2001; Hofstadter, 2001; Gentner, 2003; Gentner and Maravilla, 2017). This suggests that analogy plays a key role in diverse fields such as linguistics, psychology, cognitive science, education and artificial intelligence (Gentner, 1983; Holyoak and Thagard, 1989; Hofstadter and Mitchell, 1994; Lakoff and Nunez, 2000; Hummel and Holyoak, 2003; Richland, Holyoak and Stigler, 2004; Lu, Chen and Holyoak, 2012) among others.

Furthermore, when learners are tasked with proof of mathematical statements, evidence provides through answers and assumptions, often using examples (Minggi, Arwadi and Sabri, 2021). This experimental learning design enables students to build representations through experiences, showing proof of ability. These results have theoretical implications of educational and psychological literature on learning by analogy and classroom mathematics instruction. Drawing connections and comparing representations is core to mathematical thinking and generalizable learning (see National Mathematics Panel, 2008), but it is seriously underutilized in remote and border area in classroom teaching.

The research proposes that students can successfully explain proof using language which connects and supports the motivation as well as justification of mathematical efforts of the learners (Cobb, Yackel and Wood, 2020). Proof is considered as a form of justification, though not all justifications qualify as proof. According to Buchbinder & Mc Crone (2020), students learn by justifying mathematical ideas in proof and marking the initial stages of learning. When given structures mirror etymological sense, the representation becomes a model of thinking, valid in the technical sense used (Duval, 2017). The distinction between the syntactic-semantic and the etymological sense was leveraged to provide an account of procedural-conceptual knowledge and proof validation. This distinction is similar to the one proposed by Halford and Wilson who employed category theory to develop a theory of cognitive development (Navarrete and Dartnell, 2017). They pointed out that representations in thought must be general so that they can be transferred to situations not previously experienced and argued that representations in the form of relational knowledge are necessary. They described how symbol systems and environmental elements must be set in structural correspondence by building a formal model as a symbol system. For example, developing a representation model by evaluating an equation such as $x + y = 3$ in R confirms the existence of proof.

Discussions regarding proof might be conducted a bit confusing Czocher and Weber (2020), however, showing approximately also requires expressing in writing, using a style influenced by individual preferences and traditional learning methods. As shown by Sari, Kartowagiran and Retnawati (2020), the formal instruction of proof persists until an individual reaches a high level of proficiency. Following this, it is crucial to acknowledge that constructing an argument includes three types of available information namely (a) what is given, (b) what is being asked, and (c) the feedback received. The composition requires a specific approach to how information is presented and organized. Furthermore, research conducted by Davies, Alcock and Jones (2020) and Dawkins and Weber (2017) explores how learners respond to occurrences in the classroom, identifying these responses as critical factors for effective teaching.

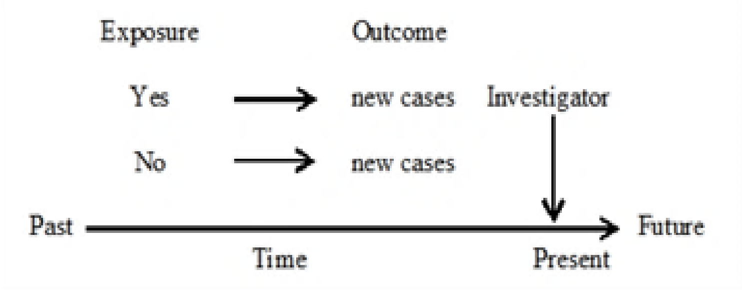
3. Methodology

We used a novel approach by bridging the etymology sense of meaning to the study of proof that identified within everyday mathematics lessons taught Internationally. The insights from the literature on proof allows for considering new strategies for drawing learners' attention to the key structural, mathematical correspondences in a learning context rather than surface features. We do not minimize the challenges in incorporating such strategies into current practices. That is classroom feasible and would not require a full re-organization of currently normative teaching practices. We argue that these pedagogical tools free learners' resources and focus their attention onto key mathematical structure. As this is a key feature presents for fostering mathematics learning.

Research Design

The research design is a meta-pattern mapping, where information (Leary and Walker, 2018) was extracted from variables and various types were analyzed in a long-term retrospective result conducted by Talari and Goyal (2020). The exploration recalled a specific point of proof problems of subject and, identified individuals with exposure to etymology as well as those without, and the prospective aspect could be seen in Figure 1.

Figure1: The Retrospective Design of the Research



The design allows the researcher to formulate hypotheses about possible associations between an outcome and an exposure and to further investigate the potential relationships. We use academic databases and the method is for a rare outcome, using Robust statistics as an approach to parameter estimation in the degree to which they are affected by violations of model assumptions.

To identify ability at the onset, words (a semiotic type of thinking in proof) were used, and an increasing trend was observed. Cases related to proof performance results were examined (Zhou and Bao, 2009; Hakim and Murtafiah, 2020). Moreover, performance was analyzed, at two dimensions of a graph axis based on etymological understanding and in meta-pattern quadrants.

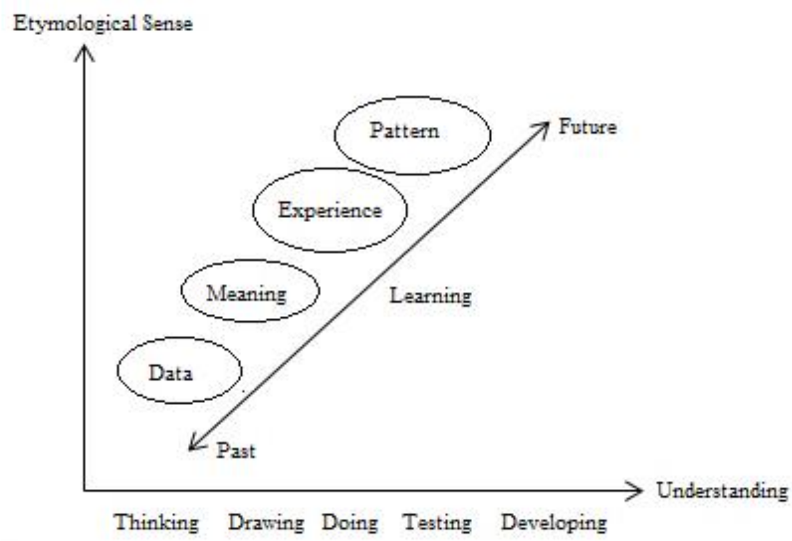
The pattern was used to integrate etymological sense of meaning into mathematical proof, organized into four quadrants. The x-axis ranged from velocity to viscosity, while the y-axis extended from monophonic (or word texture) to context. The intersection of monophonic and viscosity was the information cell, as well as the intersection cell of velocity and context axes (Cai et al., 2022). In addition, this was used to measure the learning effect (Liu, Wang and Yuan, 2022). The fundamental idea was that the ability of any student included a change in etymology (Nielsen et al., 2022).

Sample and Data Collection

We take a simple random sample with a sampling 10% from a population of 1,500 students that have a probability of being selected into the sample, i.e. of 150 observations. We use Stratified Sampling based on the students' academic year (stratum) of 2018, 2019, 2020, 2021, and 2022 of the proporsionate. The sample size of each stratum of k is $N_k = \frac{P_k}{P} N$, where Pk is the sample size and P is the population size, Nk is the proporsionate sample of stratum k, and N is the ralative measure. The relative measures are 100, 200, 300, 400, and 500, so each of the proporsinate size is 32, 28, 29, 30, and 31, taking randomly at 2020.Insert your text here. Insert your text here. Insert your text here. Insert your text here. Insert your text here. Insert your text here. Insert your text here. Insert your text here. Insert your text here.

The research questions were obtained from the experiential review and word trajectories (Rif'at, Sugiatno and Yundari, 2022). The innovation was in the result design, describing an oscillation model as shown in Figure 2. Following this, the axes for understanding proof were based on mathematical proof, including thinking, drawing, doing, testing, as well as developing. Etymological sense axes focused on the usage of verb words in proof.

Figure 2: Design of Collecting Data



Analyzing of Data

Firstly, the exploration of meaning was conducted, and then suitable mathematical knowledge and understanding were acquired, measured with scores on an interval scale. We have focused on designing that are robust to violations of normality, due to both the frequency of nonnormality and its unwanted impact assumed normal. Secondly, examiners related any data in words or phrases to create a pattern of mathematical proof based on scores. Moreover, the intersection of understanding and the use of words was analyzed according to trends for future mathematical proof. And, the Plot Diagram was used to examine how variables interacted using the ANOVA test.

4. Findings

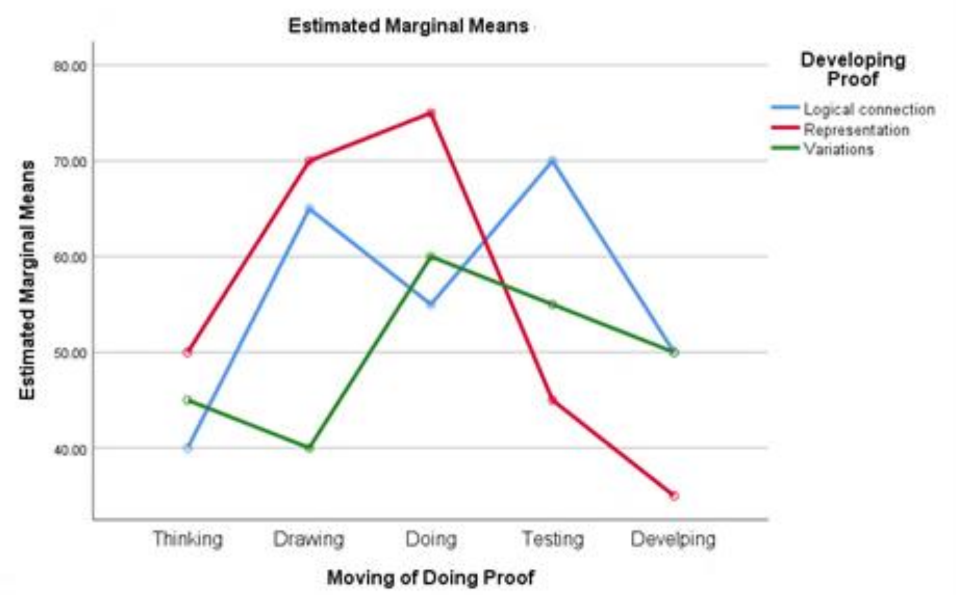
The growth trajectory across different situations is explained in etymological sense as Table 1 provided an understanding or description of the interaction. The collective impact of all independent variables (Developing Proof, Moving of Doing Proof, and the interaction) on the dependent variable (Mean Test Scores) with Significance (Sig.) < 0.05 (Alpha) affirmed the validity of the model. The intercept is also significant < 0.05 and the effects of Developing Proof as well as the Moving of Doing Proof were also significant. The multiple determination value for all independent and dependent variables is 1, signifying a strong correlation.

Table 1: The Oscillation Model of the Respondents

Developing Proof	The Moving of Doing Proof (Mean Score)				
	Thinking	Drawing	Doing	Testing	Developing
Logical connection	40	65	55	70	50
Representation	50	70	75	45	35
Variations	45	40	60	55	50

The Plot Diagram indicates a significant difference or accepting H1 as show in Figure 3. The finding proposed a connection in the development of proof based on the process of doing proof with the Significance (Sig.) < 0.05 (Alpha) affirmed the validity of the model. The intercept is also significant < 0.05 and the effects of Developing Proof as well as the Moving of Doing Proof were also significant. The multiple determination value for all independent and dependent variables is 1, signifying a strong correlation.

Figure 3: The Interaction Plot of Two Independent Variables



Students actively worked on constructing meaningful proof, empowering learners to gather information, develop arguments, and articulate ideas. Learners also acquired the ability to formulate claims and arguments in line with established norms, having access to these norms. The data is presented in Table 2, showing the trend of each etymological pair concerning the measured knowledge across all learners.

After multiple attempts of observation, certain routine problems are proved through testing with various methods that had not been confirmed yet. However, the truth is confirmed in numerous instances, eliminating any uncertainty and making the proposition appear convincingly shown. The students imagined a situation and visualized it but not yet confirmed. The test is explained by replacing the false part with a slightly modified one. Following this, the thought experiment includes inserting a single word requiring a trifling observation.

Exploration is made into techniques of students in making and verifying proof, with a focus on sense-making. There are various approaches to prove, some through equations, others by providing examples to confirm the points, to communicate the idea by exploration and analysis of objects. Proof is also achieved through trial and error for understanding about mastering facts and procedures. They also prove by advancing reasoned ways making the students' beliefs regarding as an integral part of the course. The experiences of students are viewed as an exciting domain for making sense of learning. However, the majority are unaware of basic mathematical proof strategies, specifically in Algebra, which confirms to be particularly interesting.

Table 2: Proof Model According to Etymological Sense of the Meaning

No.	Etymological Sense	Knowledge	Frequency	Understanding (the Mean Score)
1	To give an example	Procedural	100	81.50
2	To judge	Relations	87	77.00
3	To try (and or error)	equations	90	79.50
4	To verify	Operations	85	74.50
5	To determine	Functions	92	75.00
6	To test/to classify	Cases	49	71.50
7	To manipulate	Symbols	112	70.00
8	To put into	Logic	60	65.00
9	To make the equivalency	Reasoning	62	60.50
10	To finish	Algorithms	120	80.00
11	To categorize	non-constructive proof	58	55.00
12	To quantify	Congruence	61	60.50
13	To imagine	definitions	65	65.00
14	To interpret	Properties	55	57.00
15	To consider	Identity	72	70.00
16	To elaborate	Concepts	53	71.50
17	To re-express	uniqueness	60	60.00
18	To draw (to visualize)	representation	98	60.00
19	To scratch	Combining	62	60.00
20	To use words	usage and necessary symbols	74	50.00

The trajectory is in contextual measures drawn from 400 proof and categorized based on the consistency with the problems as well as divided into etymological sense. The information is summed up in Figure 4 through a diagram, and any data in words or phrases is observed to trace the trajectory of mathematical proof based on the scores.

All performances in proof construction show an increasing trend, with the initial stages signifying flexible representation and showing slight consistency (reasoning validity). The content of etymological sense includes learning experiences and information used. For example, the students used a specific number, but there is no logical connection or investigation of cases.

The results do not eliminate the possibility that high-achieving students have a better experience in mathematics teaching, including through etymological sense of meaning. The relationship proposes the possibility that students with strong proof ability tend to be more perceptive of high-quality mathematics teaching compared to other factors, such as textbooks or formal representation. In this situation, proof analysis of students shows a significant increase in formal proof construction, often disengaging from learning activities when preparing for examination problems. Students with lower increases in performance are motivated to pursue high-performance proof, study textbooks, and engage in proof activities. The learners face challenges with mathematical proof early on, specifically in logic, abstraction, as well as deductive viewpoints, and are worsened during the courses. Verbal expressions of students are frequently challenging as well as inconsistent, and in interpreting data, the use of words is strictly examined.

Figure 4: The Word Trajectory in Contextual Measure

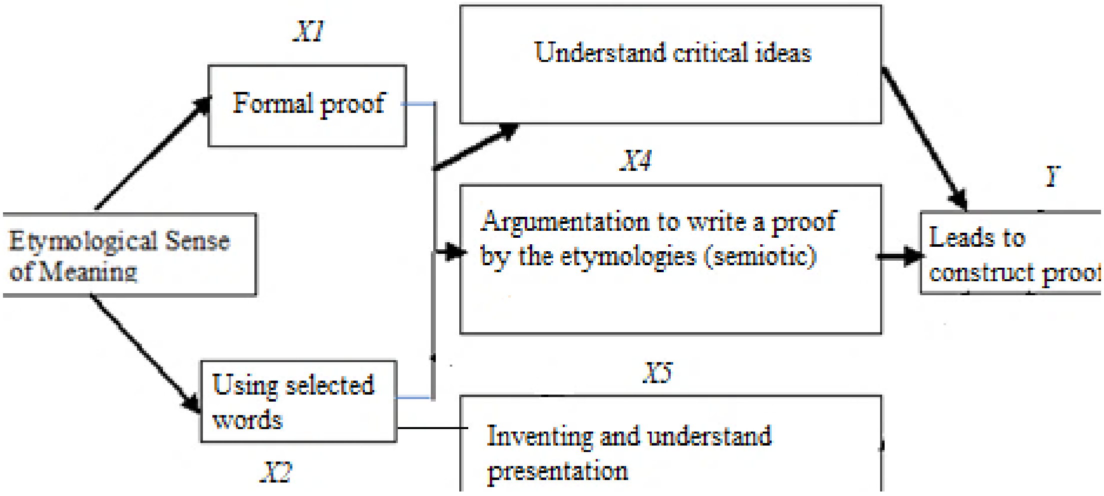
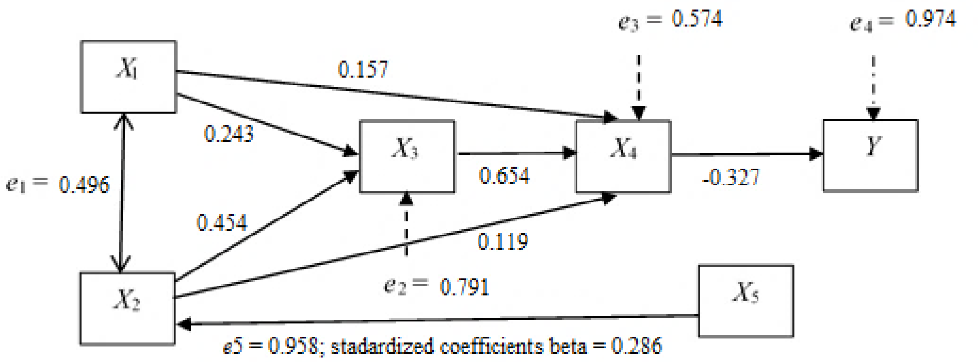


Figure 4 shows the analysis results, indicating the trajectory path that represents the actual choices made during the proof process. This path shows the connection between a potential choice and the successful construction of proof. Following this, information and knowledge are shown to students in various word contexts. For instance, when students show a symbolic implication statement, students engage in assessing, deciding, evaluating, and considering the truth. Another aspect of the performance of students includes testing and finding proof in the area of meaning. However, after approval, a comparison is made through solution design using words.

Students pass through the following steps to construct knowledge which includes (1) forming a meaning, consisting of data parts, (2) beginning to attribute meaning to these parts (data) and arranging them into a representation, (3) contextualizing as well as adding to it through experience, and (4) connecting to a pattern. The model provides a rationale for re-evaluating this trend (path analysis). The model of review with α = 5% (data in attachment).

The variables are presented in standard score form (z scores) and the model, the path coefficient equaled the observed correlation, which is decomposed into direct effect, indirect effect, unanalyzed, and spurious. The specific path coefficients are $\boldsymbol{p_{21} = 0.496, p_{31} = 0.442, p_{32} = 0.583, p_{41} = 0.112, p_{42} = -0.279, p_{43} = 0.753}$, ***and*** $\boldsymbol{p_{Y4} = 0.538}$. In path language, e signified causes outside the model, and X5 is not in the model, it only affected X2. The observation that $\boldsymbol{p_{21}} > \boldsymbol{p_{31}}$ shows the formal proof has a more significant impact on understanding critical ideas compared to using selected words. However, inventing and understanding presentation has a weak effect on understanding critical ideas and showed no relation to formal proof. In general, the model is significant in describing etymological sense of proof.

Figure 5: Model of Review of the Etymology of Proof



4. Discussion

The finding that the future shows significantly better performance in logical connection compared to different proof construction methods provides some support. However, there is no significant difference among the developing aspects. The results show improved performance due to learning experiences, with minor changes observed in the influencing factors. This result could be attributed to the initial higher cognitive demands of working with proof, requiring simultaneous processing of more information held in working memory.

The results proposed that the preference for using words in the concept of proof is favored. Words provide a more comprehensive representation than symbols, combining both numeric data and a qualitative impression.

Consequently, a word-based (semiotic) proof shows greater efficiency in various aspects of understanding. The identification of connecting different proof surfaced as a crucial learning aim in mathematics education. Aside from the justification for using words, this research uncovered additional benefits. Empirical evidence shows that adopting etymological approach improves the understanding of proof. Additionally, the widely held belief that proof teaching should solely concentrate on computational efficiency is questioned. Instead, qualitative problem-solving methods are also taken into account. These results signify that teaching and learning efficiency do not continually associate, showing the importance of including less computationally efficient proof in education.

The term proof is used to describe a word experiment leading to etymologies of students, rather than relying on symbols as a guarantee of a certain proof. An interesting discovery is made while a very formal approach is not adopted. The philosophy includes an evaluation of proof, without necessarily discouraging its validity. The analogy malfunctions when accepting proof as a real direction, and the failure is the criticism that requires reconsideration. Viewing proof as a thinking style implies proof is not a pathological case. However, for a word to serve as an objective criticism, a shared understanding of its meaning is essential. Agreement on the definition of the term, established naturally, might facilitate this and some logicians refer to knowing the extension of the concept of proof.

Nine performance variables influence the quality of teaching and learning mathematical proof. Each improvement is seen as an opportunity to explore etymology-based meanings and discover diverse proof representations. Additionally, this activity supports sense-making as well as the proof process, and the dissociation accounts of reasoning impact the performance of students.

The approach of students to proof becomes a focal point for assessing attention in mathematics education, particularly in learning and teaching mathematics. For instance, findings show that etymological sense in learning to confirm affects achievement. Attitudes toward mathematics develop early, influencing mature views of students on the subject. Unfortunately, it seems that, in learning mathematics, students still imitate from educators, textbooks, or other sources.

5. Conclusion

The paths of word meanings show significant growth, influencing the development of ideas. In the model (referred to in Figure 5), a direct effect is observed, proposing that mathematical proof of a student is considered true only when supported by intuitive understanding. However, in etymological sense, it provides a clear meaning, enabling a more comprehensive representation. When replicating sentence structures, the etymological aspect becomes a potent tool for expressing spontaneous ideas, ensuring accuracy across various contexts. Sentences including equality and those connected to the sentences represent properties relevant to subsequent expressions. The combination of two sentences addresses those not included in a formal proof. Moreover, the model supports both the sense of meaning and the completeness of proof implied through diverse representations.

The results provide understanding for teaching and learning to comprehend, including (1) initiating mathematical proof ability at etymological sense of meaning, (2) improving mathematical proof ability by thoroughly examining sentences of students, and (3) starting from sentences (or words) and gradually declining in mathematical proof development. Some learners encounter challenges, while others improve performance as well as development, and the growth, specifically in a specific manner, leads to formality. It is concluded that the performance of students varies, from using data to understand patterns to understanding etymological sense. Following this, students aim to provide formal proof, but etymological sense is crucial due to cultural learning and literacy experiences.

The way proof is developed impacts students through exercises in many textbooks, meeting the needs of learners. For proof literacy, students emphasize the importance of etymological senses progressing gradually toward formal presentation. Regarding teaching materials, both students and lecturers face challenges with the conventional approach found in texts. Students advocate for materials using language terms that are easy to read and understand. In terms of language, students specify a need for terms such as to show, to draw, to use, to test, to verify, to check, to give reasoning, to expose, and to explore in proof activities respectively. Therefore, there is a demand for etymological materials to improve representation.

The evaluation, grounded in a need analysis, shows that most students require practical materials and language skills to improve their competence in proof. A problem-solving approach is used to enhance students' proof abilities. In terms of the technical aspect of proof, most learners do not face significant difficulty, but it requires practice and study in class. Regarding materials, importance is placed on proofreading comprehension, tailored to contextual learning or proof problems. In terms of the type, 20% require formal proof, and 80% need an evaluation of practical application, albeit with some difficulty. Test techniques include individual and group assessments, requiring varied evaluations between formal logic and etymology.

To assist in students' proof, strategies have to be adopted for initiating proof, and in proof, an action word list plays a crucial role. In a sequential, realistic activity, the focus depends on the etymological sense of meaning. Words are used to establish relationships (even in challenging patterns), requiring steps to determine from the available information. An unknown variable is not solely for obtaining an answer but also for acquiring data in a symbol or representation. This is implicit in the data and necessitates thinking about the conditions related to a similar problem; therefore, etymological thinking is essential in describing proof.

6. Recommendations

This research considers learning efficiency as a characteristic of a form of etymological sense concerning the minds of students. Students with rather low cognitive capacity might be overwhelmed by multiple representations of proof. Therefore, it is worthwhile to experimentally explore the interaction between cognitive capacity and learning benefits. Furthermore, this research focuses on the effects of learning proof with different forms of action words. A follow-up research could have explored these results through developmental research.

The sense of meaning was only one of several forms of proof representation and relationships. Results on the efficiency of learning with other forms of proof representation (e.g., symbolic, verbal descriptions, real-life situations) and in other situational contexts need to be conducted. Although explorations exist on translating proof into action, future results should explore which particular aspects of functional proof could be learned using the symbolic form of words or sentences to develop mathematical language.

This research proposes that future results should focus on the selection of experiential contextual measures, providing development of mathematics learning culture. The measures are mainly designed based on the etymology and beliefs of students in proof and are related to mathematics learning. For example, a set of mathematics problems could be presented using language-based, symbolic, and representations. Therefore, there is a need to elaborate on research results focusing on mathematics education or experimental explorations in various cultures.

7. Limitations

The effect of the etymological sense of meaning estimates in the model is based on interventional and retrospective observational studies. They are therefore subject to biases and confounding that may have influenced our model estimates. However, the practical and theoretical effects of proof skill changes are estimated from the meta-pattern model with confirmatory validity analysis.

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