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Research Article

Black Hole Physics

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ARTICLE INFO | **ABSTRACT**

Based on the simple definition, a black hole refers to a point in space with extremely high gravity and density such that even light cannot escape from it. This gravitational force is so strong that it compresses all matter into a tiny region of space. In 1916, Albert Einstein first proposed the possibility of black holes existing through his general theory of relativity. The term "black hole" was coined a few years later in 1967 by the American astronomer John Wheeler. According to general relativity, a mass that is sufficiently compact can warp spacetime and form a black hole. The boundary from which nothing can escape is called the event horizon of the black hole. Black holes do not reflect any light, but they can be detected through their effects on surrounding matter and objects.
This paper aims to briefly explore the mathematical physics of black holes.

Keywords: *Black Hole, Chandrasekhar, Gravitational Collapse, Gravity, Hawking Radiation, Thermodynamics, Singularity*

Introduction

The term "black hole" was coined by Wheeler in 1967, although the possibility of such objects had been discussed long before. In the late 18th century, Michell and Laplace independently concluded that if the mass of a star was sufficiently large, its gravity would prevent light from escaping. Although this conclusion was based on Newtonian theory, the derived result for the size of such "dark stars" (gravitational radius) agrees with the later prediction of Einstein's theory of gravitation.[1]
In 1916, within a year of developing general relativity, Schwarzschild obtained the first exact (spherically symmetric) solution to Einstein's vacuum field equations. This solution describes the gravitational field of spherically compact objects. In addition to a singularity at the center of symmetry r=0, this solution had another singularity at the gravitational radius $r=r_s$. It was soon realized that this latter singularity was quite different from r=0. The nature of this Schwarzschild singularity remained a mystery for many years.
Many scientists contributed to resolving this issue Flamm 1916; Weyl 1917; Eddington 1924; Lemaître 1933; Einstein and Rosen 1935[2] before the complete final solution emerged. The main lesson learned from this study is that the space-time manifold with its metric, representing the gravitational field, may possess global features quite different from the flat Minkowski space-time of special relativity. In the case of a black hole, such global features are related to its topology and causal structure.
Although the solution for a (non-rotating) black hole has been known for a long time, the prevailing view was that nature could not accommodate an object as compact as its gravitational radius. This view was shared by the creator of general relativity himself and cited references. Some interest in the properties of highly compact gravitating systems was aroused in the 1930s following the work of Chandrasekhar (1931) on white dwarfs, and the seminal papers of Landau (1932), Baade and Zwicky (1934), and Oppenheimer and Volkoff (1939) which showed that neutron stars might exist, with radii only a few times the gravitational radius. The gravitational collapse of a massive star to form a black hole was first described by Oppenheimer and Snyder (1939).
For a long time, black holes were regarded as highly exotic and contrived objects. Names like "frozen" or "collapsed" stars were used by experts to describe them until the late 1960s. These terms reflect the attribute of a black hole to serve as a "grave" for matter. In 1963, Kerr discovered the solution to Einstein's equation describing the gravitational field of a stationary rotating black hole. This solution also possesses a gravitational radius, marking the location of the event horizon. The Kerr solution also has a true singularity, but if angular

[1] Barrow and Silk 1983, Israel 1987, Novikov 1990
[2] Flamm 1916; Weyl 1917; Eddington 1924; Lemaître 1933; Einstein and Rosen 1935

momentum $J \leq J_* = GM^2/C$ then this is cloaked by a second horizon and hence invisible to an external observer.
One new feature of the Kerr solution is the dragging of inertial frames induced by the rotation of the black hole. This effect distinguishes general relativity from Newtonian gravity. Near the horizon of a rotating black hole, this frame-dragging becomes so strong that any matter in this region is necessarily co-rotating with the hole. This region is called the ergosphere, and surrounds the black hole horizon. The angular momentum J of the black hole cannot exceed the critical value $J_* = \frac{GM^2}{C}$., otherwise the solution describes a naked singularity. It was later proven that in the absence of matter, the Kerr solution is the most general stationary black hole solution.
The Kerr metric is remarkably complicated, though stationary and axisymmetric. If one studies matter fields in this background, these symmetries are not sufficient to allow separation of variables in the Hamilton-Jacobi and field equations. It was therefore quite surprising when Carter (1968) discovered a new type of conserved quantities for the Kerr metric, associated with its hidden symmetries. This discovery made it possible to employ the method of separated variables for studying particle motion and field propagation in the Kerr spacetime.
Another major accomplishment of the late 1960s - early 1970s was the implementation of a global geometric approach to black hole theory which not only allowed one to give a covariant definition of a black hole but also proved several basic theorems. The classical theorems now state that 'black holes have no hair' (i.e. no external gravitational field other than mass, angular momentum and charge), that a black hole must have a singularity within, and that the black hole region cannot decrease. These results enabled the construction of a qualitative picture of black hole formation, to describe its likely further evolution and interaction with matter and other classical physical fields.
In 1967, in a public lecture, Wheeler coined the name 'black hole'. This name was immediately accepted with great enthusiasm by everyone. It beautifully reflects the striking features of the object.
After the discovery of pulsars (neutron stars) in the late 1960s, astrophysicists were forced to consider the prospect of observational identification of black holes. One certainly cannot see inside the gravitational radius what happens there. But the gravitational fields of black holes are so strong and possess several important distinct features that their existence can be inferred by observing the effects on matter and fields in their vicinity. For example, one could try to use the lensing effect to identify a black hole through its influence on light emitted in its neighborhood. But ordinary stellar-mass black holes have such a tiny gravitational radius (of order 10 km) that direct observation of black holes at astronomical distances by this method is practically impossible.
To search for black holes, new astronomical techniques had to be invented. Fortunately, this was also a period of remarkable innovation in astrophysics. Observational discoveries and theoretical interpretations together provided evidence for black holes and other compact stars. In 1962, Giacconi et al. (1962) identified the first extra-solar X-ray source, called Sco X-1. Such a powerful X-ray source (1000 times more intense than expected) required a new and powerful mechanism for producing X-rays. The following year, Martin Schmidt discovered that the radio source 3C 273, associated with an optically unusual spectrum source, is receding from us at high velocities (16% of the speed of light) and is therefore located 2 billion light years away. Such a distance implied it must be over 100 times more luminous than any known galaxy. Schmidt dubbed 3C 273 a "quasi-stellar" object or quasar.
New ideas were needed to power these objects. Simultaneously and independently, Salpeter (1964) and Zeldovich (1964) realized that huge bodies like black holes could release vast amounts of energy by accreting inflows of gas onto them, possibly explaining the quasar energies. They showed that the gas forms an accretion disk around the central mass which can account for the observed quasar spectra. Guseinov and Zel'dovich (1966) showed that collapsed stars in binaries could pull gas off their companion stars, which releases energy in million-degree shocks, producing an X-ray source. This idea gained importance with the discovery by Sandage et al. (1966) that Sco X-1 is associated with a blue star. Shklovsky (1967) gave a detailed theory explaining the radiation of Sco X-1 as an accreting neutron star.
The existence of neutron stars was confirmed by the discovery of pulsed radio emission by Hewish et al. (1968), and some X-ray sources were established as neutron stars by the detection of pulsations in their X-rays.[3] However, the non-detection of pulsations does not prove the existence of a black hole. More subtle methods were needed to identify them. Zel'dovich and Novikov showed that black holes do not radiate in any specific way themselves, and the identification of an X-ray source as a black hole requires measuring the mass of the compact object from observations of velocity variations of the companion star in X-ray binaries.
Webster and Murdin 1972 and Bolton 1972 provided the first evidence for the existence of the stellar-mass black holes predicted by this theory. The discovery of other types of much more massive black holes (up to 106-109 solar masses) was completely unexpected. Initially, such supermassive black holes were identified with the central engines of quasars. It is now expected that such black holes exist in the centers of many (or even most) galaxies. After these discoveries, the black hole paradigm became an important element of modern astrophysics. The gravitational field of a black hole is so strong that it may convert an appreciable fraction of the rest mass of accreting matter into radiation. This is the most powerful subsequent mechanism for

[3] Schreier et al. 1972

extracting energy from matter, second only to the annihilation of matter and antimatter. Naturally, black hole models are often invoked to explain the powerful radiations produced by highly compact objects. Another remarkable shift in the astrophysical paradigm is the realization that supermassive black holes in galactic centers may have had a profound influence on their properties, gaining importance as early as the initial stages of galaxy evolution and perhaps even the very first stages of galaxy formation.
When the unexpected new result obtained by Hawking (1974, 1975) redirected the attention of physicists back to black holes, the "news" of the possible discovery of a black hole in an X-ray binary (Cygnus X-1) was barely overshadowed. It turned out that due to vacuum instability in the strong gravitational field of a black hole, these objects are sources of quantum radiation. The most interesting feature of this radiation is that it happens to have a thermal spectrum. That is, if one neglects the scattering of radiation by the external gravitational field, a black hole radiates precisely like a hot black body. If the mass of the black hole is small (less than 1015 grams), it evaporates on a timescale shorter than the age of the universe. Such small black holes, now called primordial black holes, could only have formed in the very early stages of the evolution of the universe. (Zel'dovich and Novikov 1967, 1971; Hawking 1971) The discovery of primordial black holes or their evaporation products would provide valuable information about the physical processes occurring in that early epoch.

Hawking's discovery stimulated a large number of papers analyzing the specific features of quantum effects in black holes. In addition to the accurate description of the effects due to the creation of real particles that escape to infinity, considerable progress has been made in understanding the vacuum polarization effect in the vicinity of a black hole. This effect is important for constructing a complete quantum description of an "evaporating" black hole.
Recently, another aspect of black hole physics has become very important for astrophysical applications. The collision of a black hole with a neutron star or the coalescence of a black hole binary are powerful sources of gravitational radiation which may be strong enough to reach the Earth and be detected by the new generation of gravitational wave experiments (LIGO, LISA, and others). Detection of gravitational waves from these sources requires an accurate description of the gravitational field of the black holes during the collision phase. Essentially, gravitational astronomy opens significant opportunities for testing gravitational field theory in the realm of extremely strong gravitational fields. To do this, in addition to constructing gravitational antennas, it is also necessary to solve the gravitational equations describing this type of situation. Up to now, there have been no analytic tools available to do this. Under these circumstances, one of the important tasks is the numerical study of colliding black holes.
Half a century ago, black holes were regarded as highly exotic and contrived objects, and the general attitude in the broader physics and astrophysics communities (i.e., among scientists not working on the problem) towards these objects was quite cautious. The situation has now changed dramatically. This happened both because of new astrophysical data and due to the considerable development of the theory.

Black Hole Formation
Stellar Structure
The idea that stars are self-gravitating gaseous objects was put forward in the 19th century by Lane, Kelvin, and Helmholtz. They suggested that stars should be understood in terms of the equation of hydrostatic equilibrium.

(Equation1)

$$\frac{dp(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2},$$

(Equation 2)

$$P = \frac{\rho kT}{\mu m_p}.$$

Where k is the Boltzmann constant, μ is the mean molecular weight, T is the temperature, ρ is the mass density, and m is the proton mass. Kelvin and Helmholtz suggested that the source of heat is the gravitational contraction of the gas. However, if the luminosity of a star like the Sun is considered, the entire available energy would be liberated in ~30 million years, which contradicts geological evidence that can be found on Earth.
Arthur S. Eddington made two fundamental contributions to the theory of stellar structure, proposing that (Eddington, 1926) the source of energy is of nuclear origin, and (2) the outward radiation pressure must be included in the equation. The basic equations for stellar equilibrium then become:

(Equation 3)

$$\frac{d}{dr}\left[\frac{\rho kT}{\mu m_p}+\frac{1}{3}aT^4\right]=-\frac{GM(r)\rho(r)}{r^2},$$

$$\frac{dp_{\text{rad}}(r)}{dr}=-\left(\frac{L(r)}{4\pi r^2 c}\right)\frac{1}{l},$$

(Equation 4)

$$\frac{dL(r)}{dr}=4\pi r^2\varepsilon\rho,$$

Where l is the mean free path of photons, L is the luminosity, and ε is the energy generated per gram of matter per unit time.

When the nuclear fuel of the star is exhausted, the radiation pressure contribution drops dramatically with decreasing temperature. The star then contracts until a new source of pressure counterbalances the gravitational attraction: the degeneracy pressure of electrons. The equation of state for a degenerate electron gas is:

(Equation 5)

$$p_{\text{rel}}=K\rho^{4/3}.$$

Then using Equation 1 we will have:

(Equation 6)

$$\frac{M^{4/3}}{r^5}\propto\frac{GM^2}{r^5}.$$

Since the radius cancels out, this relation can be satisfied with a unique mass:

(Equation 7)

$$M=0.197\left[\left(\frac{hc}{G}\right)^3\frac{1}{m_p^2}\right]\frac{1}{\mu_e^2}=1.4M_\odot,$$

Where μ_e is the mean molecular weight of electrons. The result indicates that a completely degenerate star has this mass. This limit was found by Chandrasekhar (1931) and is known as the Chandrasekhar limit.

In 1939, Chandrasekhar conjectured that massive stars could produce a degenerate core. If the degenerate core reached sufficient density, protons and electrons would combine to form neutrons. This would cause a sudden decrease in pressure, leading to the star's collapse into a neutron core and the release of an immense amount of gravitational energy. This could be the origin of the supernova phenomenon." (Chandrasekhar 1939) The implication of this prediction was that the masses of neutron stars (supported by degenerate nucleon pressure) should be close to the maximum mass of white dwarfs. Shortly before, Baade and Zwicky had stated: "With all deference to the unknowable, we advance the view that supernovae represent the transition from the ordinary stars into neutron stars, which in their final stages consist of extremely closely packed neutrons." In a seminal paper (Baade and Zwicky, 1934), they not only invented neutron stars but also proposed a theory for supernova explosions, suggesting further that these explosions are the source of cosmic rays.

In the 1930s, neutron stars were not regarded as a serious physical possibility. Oppenheimer and Volkoff (1939) concluded that if the neutron core grew too large, "...either the Fermi equation of state would have to break down at very high densities, or the star would continue to contract indefinitely and never attain equilibrium." In a subsequent paper, Oppenheimer and Snyder (1939) opted for the second alternative: "When all thermonuclear sources of energy are exhausted, a sufficiently heavy star will collapse. This contraction will never quite attain the limiting radius...called the gravitational radius. Light will forever be confined within this radius. What was previously called a 'neutron star' would then have to be redescribed as a gravitational field of intenser interest." This redescribed entity was later referred to as a black hole. The scientific community paid little attention to these results, and Oppenheimer and many other scientists concentrated their efforts on the more immediate crisis of winning a war.

Stellar Collapse

Black holes form whenever matter and fields are compressed beyond their respective Schwarzschild radii. This can happen in various ways, from particle collisions to stellar explosions or the gravitational collapse of dark matter in the early universe. The most common mechanism for black hole formation in our galaxy appears to be gravitational collapse. An ordinary star remains stable as long as the nuclear reactions occurring in its interior generate enough thermal pressure to support it against gravity. Nuclear burning gradually transforms the stellar core from H to He and in the case of massive stars further to C and eventually to Fe. The core contracts during this process to reach ignition temperatures for each stage of thermonuclear burning.
Ultimately, the endothermic collapse of the iron-group nuclei, which have the tightest binding, transforms the core collapse into that of a stellar-mass black hole. Stars in the mass range 20-30Msun produce black holes with 5-15Msun. Low-mass 3-20Msun black holes can result from the collapse of stars accompanied by the explosive ejection of the outer stellar layers in an event known as a Type II supernova. A similar event, occurring in 8-20Msun stars, leaves behind a neutron star. Very massive, rapidly rotating stars are likely to result in a gamma-ray burst and the formation of a highly spinning massive black hole. Binary stellar systems evolve differently.
In Figure 1 we show an Eddington-Finkelstein diagram of the gravitational collapse of a star. A black hole forms when the future light cones tilt over to become aligned with the time axis: light rays can no longer escape to the outer world. Various paths that can lead to a stellar-mass black hole are shown in Figure 2.

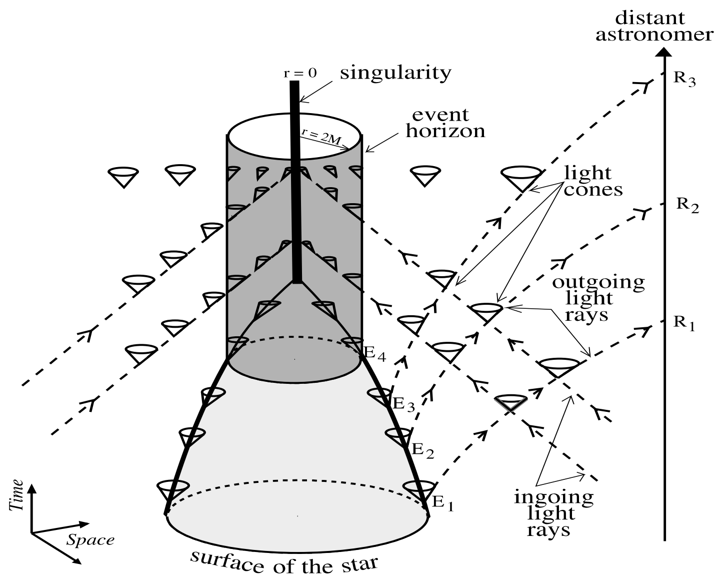

Figure 1: Eddington-Finkelstein diagram of a collapsing star with subsequent black hole formation.

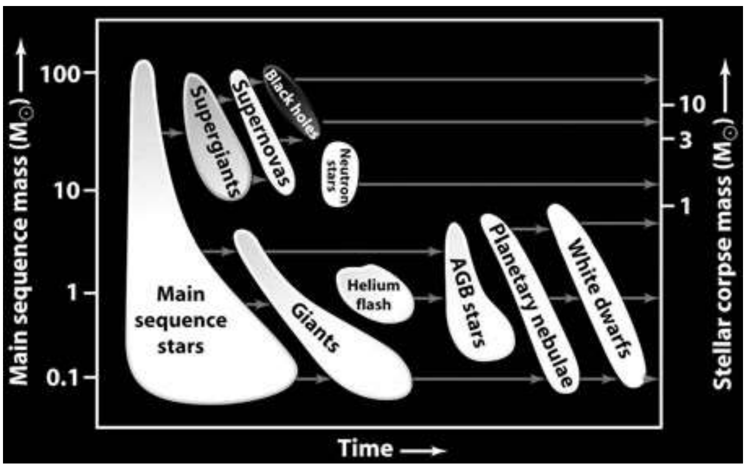

Figure 2: Stellar life cycle and channels for black hole formation.

If the collapse is not perfectly spherical, any asymmetry in the resulting black hole radiates away as gravitational waves, so that the end state is a black hole completely characterized by just the three parameters M, J, Q. The black hole retains no information about the details of the formation process or its previous history.

Gravitational collapse can also result from inhomogeneities in the initial metric, leading to the formation of small black holes as Hawking (1971) suggested, although the abundance of microscopic black holes is severely constrained by observations of the cosmic gamma-ray background radiation.
The stellar matter is described by an equation of state P=P(ρ). If as a first approximation the star is modeled as a spherical dust cloud, as done by Oppenheimer and Snyder (1939), the interior is pressure-free: P=0 and the energy-momentum tensor $T^{\mu\nu} = \rho u^{\mu} u^{\nu}$, with ρ the energy density and uμ the 4-velocity field. Solving Einstein's field equations completely determines the metric coefficients and yields the line element similar to the homogeneous isotropic Friedmann model:

(Equation 8)

$$ds^2 = dt^2 - R^2(r,t)\left[\frac{dr^2}{1-r^2} + r^2 d\Omega^2\right],$$

where $R(r,t)$ is a time-dependent scale factor, $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$ is the metric on a 2-sphere, and we adopt units with C=1. Since the cold dust does not radiate, the exterior spacetime solution matches the Schwarzschild vacuum by Birkhoff's theorem. The interior and exterior vacuum solutions must be joined on the surface of the collapsing cloud. When the collapse is complete, the final spacetime becomes Schwarzschild.
A general formalism for spherically symmetric gravitational collapse including pressure was developed. The energy-momentum tensor is now $T^{\mu\nu} = (\rho + P)u^{\mu}u^{\nu} + Pg^{\mu\nu}$ and the line element in comoving coordinates is (Misner and Sharp 1964; Joshi 2007):

(Equation 9)

$$ds^2 = e^{2\varphi}dt^2 - e^{\lambda}dr^2 - R^2(r,t)d\Omega^2,$$

where φ and λ are functions of r and t. The 4-velocity components are $u^0 = e^{-\varphi}, u^i = 0,\ for\ i = r, \theta, \phi$.

We can now introduce the mass function m defined by:

(Equation 10)

$$e^{\lambda} = \left(1 + \dot{R}^2 - \frac{2m}{r}\right)^{-1}\left(\frac{\partial R}{\partial r}\right)^2,$$

Where the dot denotes differentiation with respect to t and multiplication by $e^{-\varphi}$.

(Equation 11)

$$\dot{f} = u^{\mu}\frac{\partial f}{\partial x^{\mu}} = e^{-\varphi}\left(\frac{\partial f}{\partial t}\right).$$

This is the proper time derivative in the comoving frame.

Combining the conservation law $T^{\mu\nu} = 0$ and solving Einstein's field equations, we can find the Misner-Sharp equations for spherical symmetric collapse:

(Equation 12)

$$\dot{m} = -4\pi R^2 P \dot{R},$$

$$\ddot{R} = \left(\frac{1 + \dot{R}^2 - 2m/r}{\rho + P}\right)\left(\frac{\partial P}{\partial R}\right) - \frac{m + 4\pi R^3 P}{R^2},$$

$$\frac{\partial m}{\partial R} = 4\pi R^2 \rho.$$

These equations, together with the equation of state relating P and ρ, determine the dynamical evolution of homogeneous spherical stellar collapse. For P=0 the Oppenheimer-Snyder (1939) results are recovered. When P≠0 the solution requires numerical integration. Any solution requires specifying initial data for R(r,0), m(r,0) and U(r,0) with $U = e^{-\varphi}R$ also needed so that the functions R,m,U all remain regular at r=0. If r_b denotes the outer boundary of the matter distribution, $m(r_b, t) = M$ and the interior metric can be smoothly joined at r= r_b to the exterior Schwarzschild metric with mass M.

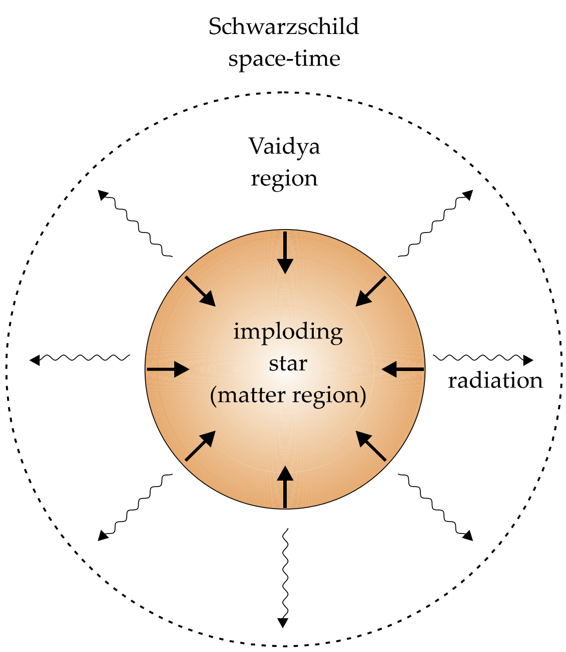

Figure 3: A radiating collapsing star and the different regions associated with the collapsing matter, radiation field, and exterior vacuum spacetime.

The analysis mentioned above does not take into account the effects of the radiation field of the collapsing star. If radiation is emitted by the star, the exterior solution is not a vacuum Schwarzschild solution. Although the radiation effects of an ordinary star on the spacetime metric are negligible, it can become important in the later stages of gravitational collapse. A radiating star undergoing collapse will be surrounded by an outward propagating region of radiation. Only far from the radiation region can spacetime be described by the Schwarzschild solution. Thus, the system comprises three regions: the collapsing spacetime interior, the radiation region, and the exterior Schwarzschild spacetime, which asymptotically approaches Minkowski spacetime. (Figure 3)
To find the metric in the radiation region, Einstein's field equations must be solved for an energy-momentum tensor of the form:

(Equation 13)

$$T_{\mu\nu} = \sigma k_\mu k_\nu ,$$

where k_μ is an outward pointing null vector, and σ is the radiation energy density in a locally Lorentz-boosted frame moving with velocity u_μ.
A solution of this kind was found by Vaidya (1943, 1951). In outgoing null (u, r, θ, ϕ), the metric is given by the line element:

(Equation 14)

$$ds^2 = \left(1 - \frac{2m(u)}{r}\right)du^2 + 2du\,dr - r^2 d\Omega^2,$$

where m(u) is an arbitrary non-decreasing function of the retarded time coordinate u. The second term relates to the Schwarzschild time t:

(Equation 15)

$$u = t - r - 2m\log(r - 2m).$$

From Eqs. 13, 14 and the Einstein equations

(Equation 16)

$$\sigma = -\frac{1}{4\pi r^2}\frac{dm(u)}{du}.$$

The total luminosity of the star at infinity is $L = -dm(u)/du$. Hence m represents the mass of the system and the energy flux is the rate of decrease of m. The properties of the Vaidya spacetime were discussed by Lindquist et al. (1965). In particular, they showed that, unlike the Schwarzschild case, the surface *r = 2m* is a null surface. No timelike or null geodesic from the exterior can penetrate it to reach the interior region. This is illustrated in Figure 4.

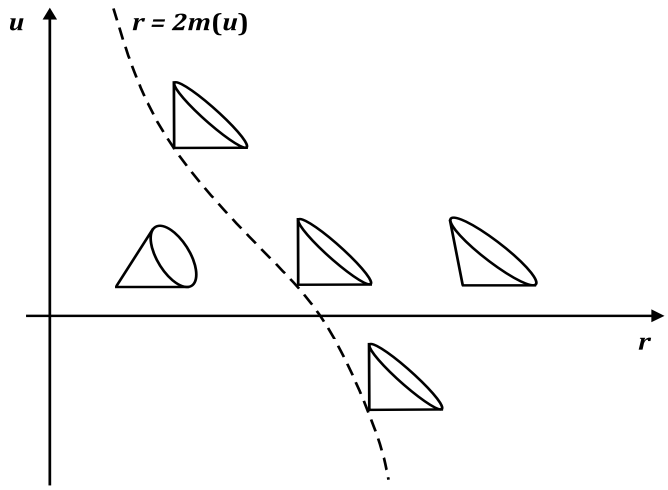

Figure 4: Light cones for the Vaidya metric. The surface r = 2m is a null surface.

An interesting aspect of the gravitational collapse of a radiating star is that for certain choices of the mass function m(u) (e.g. m(u) = M(1 - e^-λu) where the rate of collapse is controlled by the parameter λ), it may be possible to have a "slow" collapse (by taking λ small) such that the formation of an event horizon is delayed beyond the formation of the singularity, actually allowing outgoing null geodesics originating close to the singular region to escape to infinity. Such effects can also be obtained with inhomogeneous collapse (see Figure 5 for a schematic of the situation). Whether singularities produced by gravitational collapse can actually be observable to external observers remains an open question.

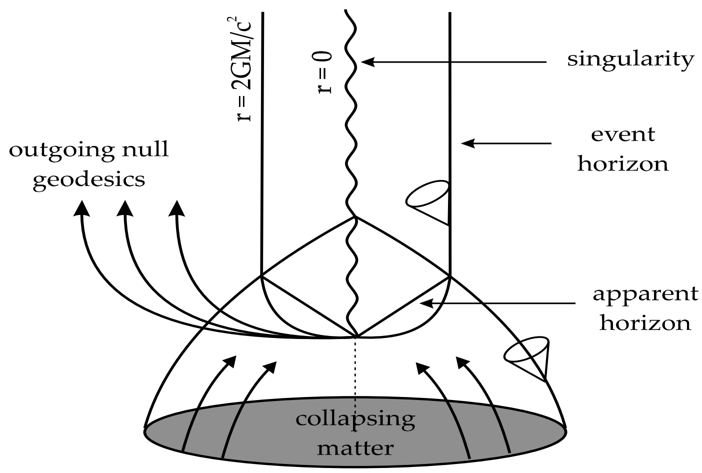

Figure 5: Formation of a black hole with an initially observable singularity. Compare with Figure 1 where the horizon always cloaks the singularity.

Supermassive Black Holes

Supermassive black holes can arise from various processes occurring at the centers of galaxies discussed by Rees (1984). See Figure 6 for a schematic of some of the pathways. However, some current views indicate that galaxies may have formed around pre-existing seed supermassive black holes resulting from the gravitational collapse of dark matter.

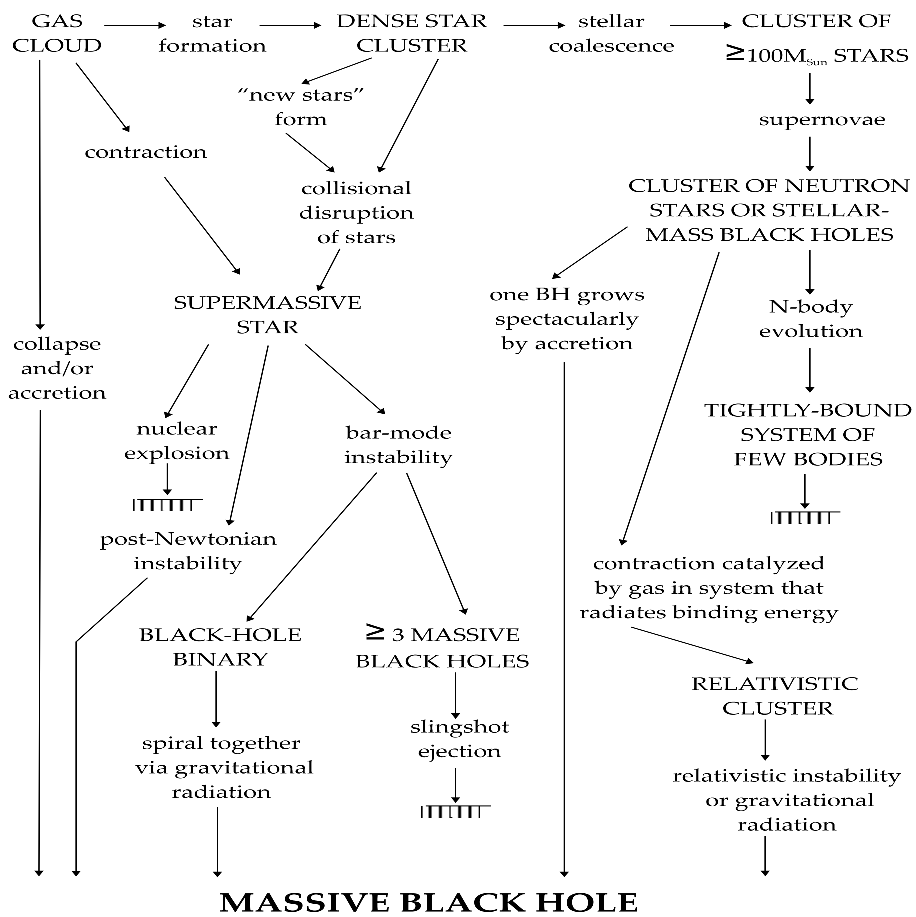

Figure 6: Channels for the formation of supermassive and ultramassive black holes.

Dynamical evidence confirms the existence of ultramassive black holes with masses ranging from $\sim 10^6$ to $\sim 10^9 M_\odot$ at the centers of most nearby galaxies. In our own Galaxy, the black hole has a mass of $\sim 4 \times 10^6 M_\odot$. The existence of black holes with masses up to $\sim 10^9 M_\odot$ in the nuclei of active galactic nuclei (AGN) has been inferred from their cosmological redshifts $z > 6$. Less massive, but nevertheless very massive, black hole populations may have existed in large numbers in the early, dark ages of the universe after the first recombination of protons and electrons (Volonteri, 2010). Such primordial black hole formation may have resulted from one or more of the following processes: (1) gravitational collapse of the first generation of stars - the so-called Population III stars, (2) dynamical instabilities of gas, (3) stellar-dynamical instabilities, (4) collapse of dark matter, and (5) initial density perturbations.

Population III stars form from zero-metallicity gas after the first recombination. These stars are expected to be very massive $M_* 100 M_\odot$. Massive stars evolve rapidly and at the end of their lives undergo complete gravitational collapse, leaving behind black hole remnants in the mass range from $40 M_\odot$ perhaps as high as $1000 M_\odot$, except in the $\sim 140 - 260 M_\odot$ range where pair-instability supernovae leave no compact remnants (e.g. Fryer et al. 2001). Galaxies may have formed around these primordial black holes following some of the pathways shown in Figure 6.

If fragmentation can be avoided in primordial giant clouds and cooling proceeds gradually, the gas contracts until rotation can halt the collapse. In such a case, global dynamical instabilities, such as bar-mode instabilities (Begelman et al. 2006), can shed angular momentum outwards, allowing the core to continue collapsing. The accumulated gas at the center may then build up a very massive nuclear object. Eventually, complete collapse may ensue, creating a black hole of mass $\sim 10^4 M_\odot$ or even higher.

On the other hand, fragmentation and star formation within a collapsing gas cloud can lead to the buildup of a very dense nuclear star cluster. Repeated stellar collisions can then lead to the formation of a $\sim 10^2 - 10^4 M_\odot$black hole (Devecchi & Volonteri, 2009). This process was essentially predicted by Begelman & Rees (1978).

The direct collapse of dark matter into dust continues without any heating until the rate of WIMP dark matter particle annihilation becomes significant. Then, a "dark matter star" may be supported by dark matter annihilation radiation. Of course, what is supported is the baryonic content that is dragged down by the decaying dark matter. This scenario was discussed by Freese et al. (2008), who found that these "dark stars"

have surface temperatures of 4000-10000 K, radii $\sim 10^{14}$ cm, luminosities $\sim 10^{6} L_{\odot}$ and $\sim 1000\ solar\ mass$. After the dark matter fuel is exhausted, the star collapses into a massive black hole.
Another way to produce black holes in the early universe is through initial density perturbation fluctuations. Wherever the density fluctuations are large enough that gravitational forces overcome pressure forces, complete collapse and consequently the formation of primordial black holes will ensue. The black hole masses can range from microscopic to many thousands of solar masses. However, several physical and astrophysical mechanisms limit the abundance of primordial black holes. Primordial $10^{9} - 10^{17}\ g$ black holes are highly unlikely due to unseen evaporation effects. Black holes with masses up to $\sim 10^{40}$ g $\sim 10^{7} M_{\odot}$ have been severely constrained by astrophysical effects to comprise at most a tiny fraction of the total cosmic density.

Intermediate-Mass Black Holes

As mentioned in the previous section, black holes with masses in the range $10^{2} - 10^{3}\ M\odot$ could arise from the collapse of Population III stars. More massive black holes may exist if they are fueled from galactic nuclei or commonly merge with stars or other black holes. The fueling rate from dispersed matter in the normal interstellar medium is a very slow process that produces intermediate-mass black holes (IMBHs) on timescales of order the Hubble time ($t \sim H_0^{-1} \times 10^{12}\ year$). If they exist, they must arise from interactions in a dense cluster (e.g. Miller 2003), an extremely massive young star cluster, or an old globular cluster.
Young open star clusters are only tens of millions of years old and their most massive stars are still on the main sequence. The stellar sizes are large enough to allow collisions and mergers. The overall collision rate in these clusters may be enhanced by the presence of binaries. Whether the mass accretion rate allows IMBH formation is still an open question that depends on the delicate balance between the collision frequency and consequent mass buildup in the seed black hole, and the mass lost from the cluster in winds, supernovae, and other disruptive processes. So far there is no firm evidence for the existence of IMBHs in young clusters.
Globular clusters are old systems, and most of their stellar populations consist of compact remnants like neutron stars or white dwarfs. The collision cross section of such objects is negligible, so seed black holes are not expected to grow significantly by mergers. However, a substantial population of binaries is expected to exist in old star clusters. Binaries lose energy to gravitational radiation and ultimately coalesce into a single black hole. If the black hole exceeds $\sim 50 M\odot$, its inertia will cause it to sink to the core of the cluster, where repeated mergers can build it up to masses of $\sim 10^{3} M\odot$ or more within a Hubble time (Miller & Hamilton 2002). In addition, interactions between single black holes and binaries within the cluster can lead to recoil kicks that may eject black holes at velocities exceeding 50 km/s (Webbink, 1985). This may allow black hole mergers to occur outside the globular cluster when binaries get ejected.
Ultra-luminous X-ray sources have been found in a number of galaxies. These sources show X-ray band luminosities exceeding $10^{40}\ erg\ 10^{-1}$, in one case even reaching $3 \times 10^{41}\ s^{-1}$ erg/s. It has been suggested that these and similar sources may be intermediate-mass black holes (IMBHs) radiating at sub-Eddington luminosities, but finding conclusive evidence has so far proved elusive.

Smaller Black Holes

If the Planck scale involves extra dimensions, black holes with masses much less than a solar mass may form from initial density fluctuations or through particle collisions. In general, if in some region the density fluctuations are large enough that the gravitational field is able to overcome the pressure forces, the whole region will undergo complete collapse to form a primordial black hole. The mass of a primordial black hole arising from cosmic density fluctuations at time t after the Big Bang is given by: (Carr et al. 2010)

(Equation 17)

$$M \sim \frac{c^3 t}{G} \sim 10^{15} \left(\frac{t}{10^{-23}\ \mathrm{s}} \right) \mathrm{g}.$$

Black holes formed at the Planck time would have the Planck mass $\sim 10^{-5} g$, while those forming precisely $\sim 10^{23}$ second after the Big Bang would be $\sim 10^{15} g$. As will be discussed in later sections, black holes of this mass or lower should have evaporated by now. Since such black holes would have had to produce ~100 MeV photons at the present epoch, the observational bound on the intensity of the gamma-ray background at these energies indicates their density cannot exceed $\sim 10^{-8}$ of the critical density. This completely rules out small primordial black holes as a dark matter candidate. However, current data cannot rule out a significant density contribution from black holes with masses below $(10^{20} - 10^{26} g)$ and intermediate masses $(10^{2} - 10^{4} M\odot)$. Black holes evaporating after the so-called nucleosynthesis time (about 1 second after the Big Bang) have even been proposed as a source for reionization of the universe (Fang, 2002 & saito,2008)

Black Hole Thermodynamics

The area of a Schwarzschild black hole is given by:

(Equation 18)

$$A_{\mathrm{Schw}} = 4\pi r_{\mathrm{Schw}}^2 = \frac{16\pi G^2 M^2}{c^4}.$$

For a Kerr-Newman black hole, this area is:

(Equation 19)

$$A_{\mathrm{KN}} = 4\pi \left(r_+^2 + \frac{a^2}{c^2} \right)$$
$$= 4\pi \left[\left(\frac{GM}{c^2} + \frac{1}{c^2}\sqrt{G^2M^2 - GQ^2 - a^2} \right)^2 + \frac{a^2}{c^2} \right].$$

Note that Eq. (19) reduces to Eq. (18) for $a = Q = 0$.
When a black hole absorbs a mass δM, its mass increases to $M + \delta M$, and consequently its area also increases. Since the horizon is only one-way traversable, the area of a black hole can never decrease. This shows an analogy with entropy. The change in the black hole entropy is related to the heat (δQ) absorbed via the relation:

(Equation 20)

$$\delta S = \frac{\delta Q}{T_{\mathrm{BH}}} = \frac{\delta M c^2}{T_{\mathrm{BH}}}.$$

Particles trapped by the black hole will have a wavelength of order:

(Equation 21)

$$\lambda = \frac{\hbar c}{kT} \propto r_{\mathrm{Schw}},$$

where k is Boltzmann's constant. Then:

(Equation 22)

$$\xi \frac{\hbar c}{kT} = \frac{2GM}{c^2},$$

where ξ is a numerical constant. We can therefore associate a temperature with the black hole:

(Equation 23)

$$T_{\mathrm{BH}} = \xi \frac{\hbar c^3}{2GkM},$$

and

(Equation 24)

$$S = \frac{c^6}{32\pi G^2 M} \int \frac{dA_{\mathrm{Schw}}}{T_{\mathrm{BH}}} = \frac{c^3 k}{16\pi \hbar G \xi} A_{\mathrm{Schw}} + \text{constant}.$$

A quantum mechanical computation of the temperature at the Schwarzschild horizon yields $\xi = (4\pi)^{-1}$. Therefore,

(Equation 25)

$$T_{\mathrm{BH}} = \frac{\hbar c^3}{8GMk} \cong 10^{-7}\ \mathrm{K}\left(\frac{M_\odot}{M}\right).$$

and we can write the black hole entropy as:

(Equation 26)

$$S = \frac{kc^3}{4\pi\hbar G} A_{\text{Schw}} + \text{constant} \sim 10^{77} \left(\frac{M}{M_\odot}\right)^2 k\,\text{J}\,\text{K}^{-1}.$$

The formation of a black hole represents an enormous increase in entropy. For comparison, a star has an entropy of order ~20 less than the corresponding black hole. This tremendous entropy increase is associated with the complete loss of structure in the original system (e.g. a star) after collapse to a black hole.
The analogy between area and entropy allows us to state a set of laws for black hole thermodynamics (Bardeen et al. 1973):

- **First Law** (Energy Conservation): $dM = T_{BH} ds + \Omega + dJ + \Phi dQ + \delta M$. where Ω_+ is the angular velocity, J is angular momentum, Q electric charge, Φ electrostatic potential and δM is contribution to the change in the black hole mass due to the change in the distribution of external stationary matter.
- **Second Law** (Entropy Never Decreases): In all physical processes involving black holes, the total area summed over all event horizons can never decrease.
- **Third Law** (Unattainability of Absolute Frozenness): The temperature (surface gravity) of a black hole cannot be zero. Since $T_{BH} = 0$ with $A \neq 0$, for extremal charged and extremal rotating (Kerr) black holes, it is conjectured that these are idealized limiting cases that cannot be realized in nature.
- **Zeroth Law** (Thermal Equilibrium): The surface gravity is constant over the entire event horizon of any stationary, axisymmetric black hole.

Quantum Effects for Black Holes

If a temperature can be associated with black holes, they should radiate like any other body. The luminosity of a Schwarzschild black hole is given by:
(Equation 27)

$$L_{\text{BH}} = 4\pi r_{\text{Schw}}^2 \sigma T_{\text{BH}}^4 \sim \frac{16\pi\sigma_{\text{SB}}\hbar^4 c^6}{(8\pi)^4 G^2 M^2 k^4}.$$

Here σ_{SB} is the Stefan-Boltzmann constant, which can be rewritten as:

(Equation 28)

$$L_{\text{BH}} = 10^{-17} \left(\frac{M_\odot}{M}\right)^2 \text{erg}\,\text{s}^{-1}.$$

The lifetime of a black hole is:

(Equation 29)

$$\tau \cong \frac{M}{dM/dt} \sim 2.5 \times 10^{63} \left(\frac{M}{M_\odot}\right)^3 \text{yr}.$$

Note that the black hole heats up as it radiates - this happens because when the hole radiates, its mass decreases and then according to Eq. 25 the temperature must increase.
If nothing can escape from black holes due to the event horizon, what is the origin of this radiation? The answer found by Hawking (1974) involves quantum effects near the horizon. According to the Heisenberg uncertainty principle $\Delta t \Delta E \geq \hbar/2$, particle pairs can emerge out of the vacuum state as long as the violation does not persist too long. The particles must be created as pairs and for a short time in order to satisfy conservation laws other than energy. If a pair is created near the horizon and one particle crosses over, the other may escape provided its momentum is outwards directed. The escaping virtual particle then becomes a real particle at the expense of the black hole's energy. The black hole then loses energy and its area slowly decreases, violating the second law of thermodynamics. However, if we consider the generalized second law, which always holds, there is no violation: in any process, the total generalized entropy $s + s_{BH}$ never decreases (Bekenstein, 1973).

Black Hole Magnetospheres

In the real universe, black holes are not expected to be isolated, so the ergosphere must be populated by charged particles. This plasma will be dragged into rotation in the same sense as the hole due to frame-dragging effects. A magnetic field is generated and dragged along, creating a potential drop that may accelerate particles to relativistic speeds and drive a wind along the hole's rotation axis. Such a picture has been repeatedly invoked (Punsly, 2001 & Coroniti, 1990)

In Figures 7 and 8 we illustrate the behavior of fields and currents in the ergosphere. Since the whole region is rotating, the ergospheric wind arises in the direction of the large-scale magnetic field.

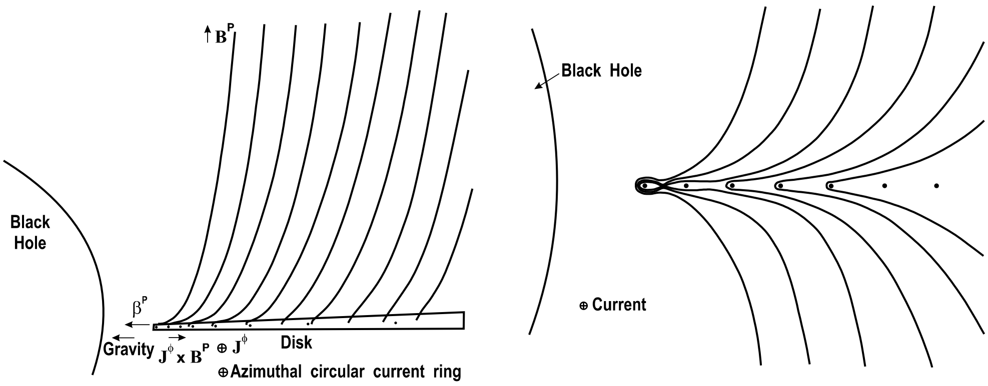

Figure 7: The inflow of matter is supported by a radial magnetic field. As the inner part of the current sheet approaches the black hole, sources are increasingly redshifted towards infinity observers and their contribution to the poloidal magnetic field decreases. At a point X the field reconnects again.

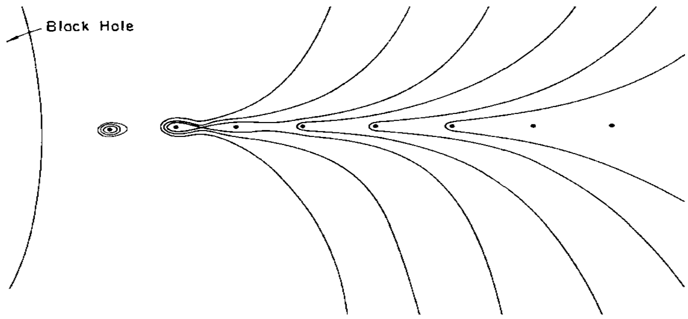

Figure 8: With continued reconnection, the magnetic field around the innermost currents becomes detached from the large-scale field, allowing for the possibility of magnetic flux destruction by the black hole.

Blandford and Znajek (1977) constructed a general theory for the steady, axisymmetric, force-free magnetosphere around a rotating black hole. In an accreting black hole, the magnetic field can be sustained by external currents, but since such currents flow along the horizon, field lines are usually depicted emerging from the horizon as a source and then being twisted up by the rotation. The result is an outgoing electromagnetic flux of energy and angular momentum. The output power is given by:

(Equation 30)

$$L \approx 10^{39}\ \mathrm{erg\,s^{-1}} \left(\frac{M}{10^6 M_\odot}\right)^2 \left(\frac{a}{a_{\max}}\right)^2 \left(\frac{B}{10^4\,\mathrm{G}}\right)^2 .$$

This picture led to the so-called "membrane paradigm" developed by Thorne et al. (1986) in which the event horizon is ascribed a set of physical properties. This model of the black hole has been severely criticized by Punsly (2001) as general relativity shows the horizon is causally disconnected from the outgoing wind. We will look at this issue further in the chapter.
Recent numerical simulations (Komissarov 2004) indicate that the key role in the electrodynamic spin-down mechanisms of rotating black holes is played by the ergosphere rather than the horizon itself. However, on large scales the Blandford-Znajek solution appears to be marginally stable. Twisted magnetic fields in the ergosphere of a Kerr black hole are shown in the figure.

Inside the Black Hole

The most striking feature of the interior Schwarzschild black hole is that the roles of space and time are interchanged: the radial direction of space becomes timelike and time becomes spacelike. Inside a spherical black hole the radial coordinate behaves like time: changes occur in the preferred direction, i.e. towards the space-time singularity. This means the black hole interior is essentially dynamical.

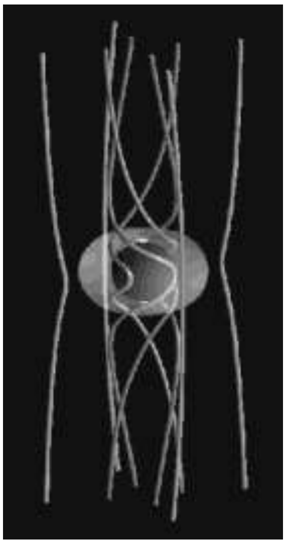

Figure 9: Effects of the Kerr black hole ergosphere on external magnetic field lines

Let us recall the Schwarzschild metric:

(Equation 31)

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

If we consider a radially moving test particle:

(Equation 32)

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2.$$

The light cone structure is defined by the condition ds = 0. Writing r_{Schw} again for the Schwarzschild radius, we obtain:

(Equation 33)

$$\left(1 - \frac{r_{\mathrm{Schw}}}{r}\right)c^2dt^2 - \left(1 - \frac{r_{\mathrm{Schw}}}{r}\right)^{-1}dr^2 = 0.$$

If we now consider the inside of the black hole $r < r_{Schw}$, then:

(Equation 34)

$$\left(1 - \frac{r_{\mathrm{Schw}}}{r}\right)^{-1}dr^2 - \left(1 - \frac{r_{\mathrm{Schw}}}{r}\right)c^2dt^2 = 0.$$

Now the sign of the space and time coordinates is interchanged. The light cones, which in Schwarzschild coordinates point along the time axis at every point outside the event horizon, now point radially inwards at every point. The path of photons is given by:

(Equation 35)

$$\frac{dr}{dt} = \mp c\left|1 - \frac{r_{\mathrm{Schw}}}{r}\right|,$$

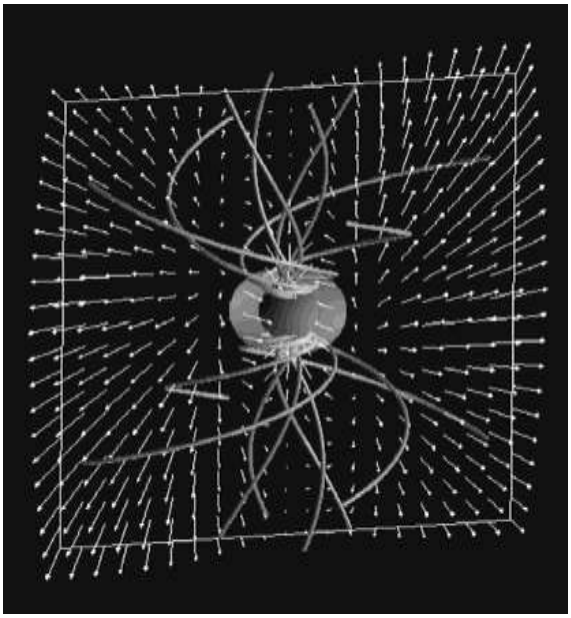

Figure 10: 3D graphic of magnetic field lines and plasma flow streamlines around a Kerr black hole. The black sphere in the center depicts the black hole event horizon. The transparent (grey) surface surrounding the black hole is the ergosphere surface. The arrows indicate the velocity vectors of the plasma flow. The tornado-like tubes show the magnetic field lines.

With r always decreasing. The light cones become thinner and thinner as r approaches r = 0 at the singularity. In addition to particle infall, there is a small flux of gravitational radiation inwards through the horizon, arising from slight disturbances outside. This radiation, in the form of material particles and photons, terminates at the singularity.

For the Kerr black hole, space and time also interchange roles between the two horizons, as happens in the interior Schwarzschild spacetime. Instead of time inevitably marching forward, the radial dimension of space inevitably marches inwards towards the second, or Cauchy, horizon - a null hypersurface beyond which predictability breaks down. Beyond this the Kerr solution predicts a second rebound in order to avoid a naked singularity and terminate in a well-behaved manner on another asymptotically flat region. In this bizarre region inside the Cauchy horizon, an observer can by choosing an appropriate trajectory loop around the singularity and travel backwards in time, encountering himself - i.e. closed timelike curves exist. Another possibility allowed by the equations for an observer in the central region is to fall through the ring singularity to emerge in an anti-gravity universe with even more bizarre physical laws. Or he could travel through two more horizons (or rather anti-horizons) to emerge at $t = -\bowtie$, in another asymptotically flat universe. All of this can be exhibited on the Penrose-Carter diagram for the Kerr black hole (Fig. 12).

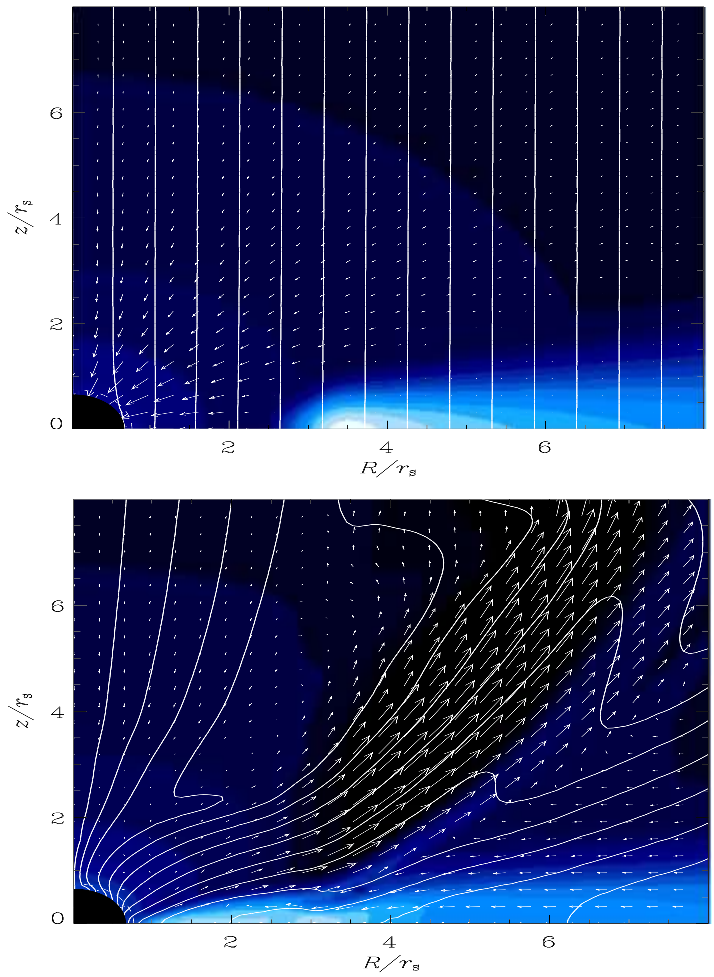

Figure 11: The black hole (corona) magnetosphere and the outflows it produces.

The above discussion about the interior of the Kerr black hole is quite academic, as in realistic black holes the inner horizon is likely unstable. Poisson and Israel (Gambini & Pullin 2011, Oriti 2009 & Rovelli 2004) have shown that when the spacetime is perturbed by a fully non-linear, ingoing and spherically symmetric vacuum thin shell, a curvature singularity of the "weak" variety develops at the inner horizon. This singularity is "weak" in the sense that none of the scalar curvature invariants actually diverges there. The development of the singularity destroys the "Kerr tunnel" leading to other asymptotically flat worlds that the inner Cauchy horizon allows. The key factor driving the instability is the infinite concentration of infalling energy density seen by an infalling observer near the Cauchy horizon. The infinite energy density arises due to the ingoing gravitational radiation that gets partially scattered by the strong interior curvature. The non-linear interaction between the ingoing and outgoing gravitational fluxes results in a weakly singular curvature at the Cauchy horizon where an exponential inflation of the mass parameter occurs. (Figure 13) This modifies the picture of the Kerr black hole interior in that instead of a Cauchy horizon acting as a veil beyond which predictability breaks down, we have a microscopically thin region near the inner horizon where the curvature is highly magnified. (Poisson & Israel 1990) Other analyses based on the study of spherically symmetric perturbations seem to indicate that rather than a weakly singular null singularity, it is a strong spacelike singularity that forms under generic non-linear perturbations (Poisson and Israel, 1990). This is the same conclusion one arrives at through a linear perturbation analysis of the inner horizon. More recent numerical studies using periodic initial data also find a null singularity of the mass inflation variety (Yurtsever, 1993)

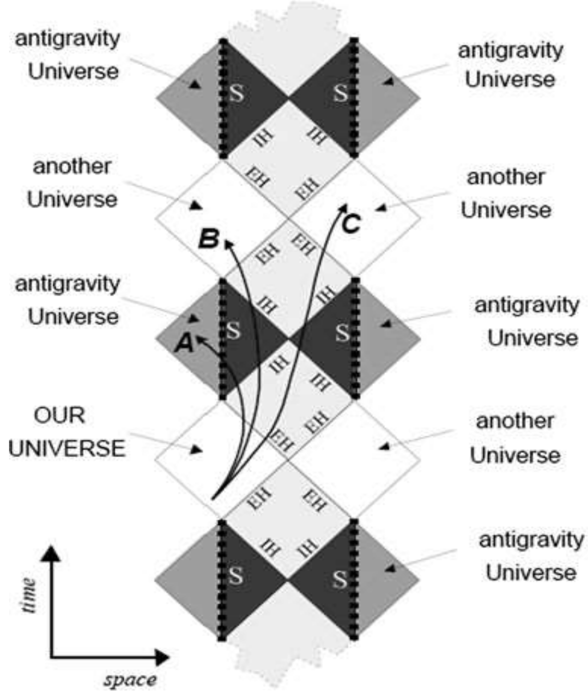

Figure 12: Penrose-Carter diagram for the non-extremal Kerr solution. The pattern repeats infinitely in both directions. One trajectory terminates at the singularity (A), two escape trajectories are shown (B and C). IH stands for "inner horizon", EH for "event horizon" and S labels the "singularity".

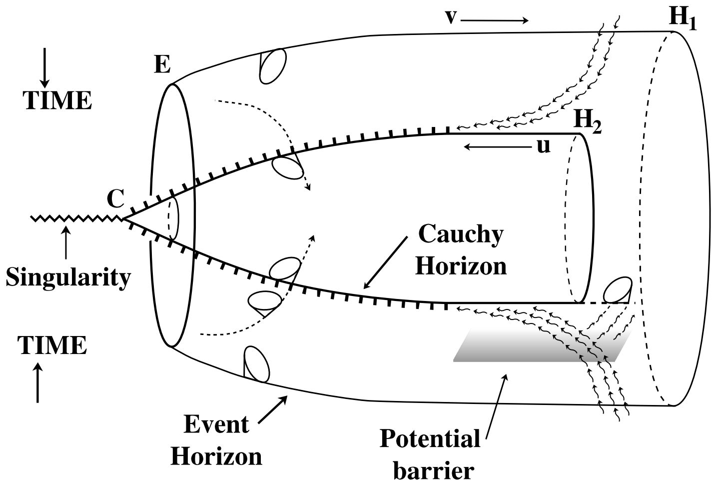

Figure 13: Diagram showing the interior of the Kerr black hole and the energy-momentum accumulation at the inner horizon. H1 and H2 are the outer and inner horizons respectively.

Singularities

A spacetime is said to be singular if the manifold M representing it is incomplete. A manifold is incomplete if it contains at least one inextendible curve. An inextendible curve $\gamma: [0, a) \longrightarrow M$ is one for which there is no point p in M such that $\gamma(s) \rightarrow P$ as $a \rightarrow \ s, i.e. \gamma$ it means γ has no endpoint in M. A given spacetime $(M, g_{ab})$is said to be extendible if there exists an isometric embedding θ : M → M', where (M',g') is a spacetime and 'ab θ is defined on some suitable subset of M. A spacetime is singular if it contains a curve γ that is inextendible in the sense described above. It is said that singular spacetimes contain "singularities", but this is an abuse of language: singularities are not "things" in spacetime, but rather a pathological feature of the theory. In fact, by definition, "singularities" cannot exist in spacetime.

The so-called coordinate singularities are not real features of a singular spacetime. It appears that the spacetime is singular in some representations, but the pathologies (divergences) can be removed by a coordinate transformation, like the "Schwarzschild singularity" at $r_{Scwh} = 2GM/C^2$in the Schwarzschild spacetime. For instance, we can change the description of the spacetime to Eddington-Finkelstein coordinates and then see that timelike geodesics can be extended through the "singular" point on the manifold. Essential singularities cannot be removed in this fashion. For example, this happens with the r=0 singularity in the Schwarzschild spacetime or with the ring singularity at r=0, θ=π/2 in the Kerr metric written in Boyer-

Lindquist coordinates. The scalar curvature $R^{\mu\nu\rho\sigma}\, R_{\mu\nu\rho\sigma}$ diverges. There is no metric defined there and the Einstein equations break down.

An essential or genuine singularity should not be interpreted as representing some physical object with infinite density, infinite pressure etc. Since the singularity does not belong to the manifold representing spacetime in general relativity, it simply cannot be described or represented within such a theory. General relativity is incomplete in the sense that it cannot provide a complete description of the gravitational behavior of every physical system. Real singularities lie outside the range of validity of the bounded variables of the theory: they do not belong to the spacetime ontology that can be described by 4D differential manifolds.
An essential singularity in solving the Einstein field equations is one of two things:

1) A situation where matter is forced to be compressed to a single point. (spacelike singularity)
2) A situation where specific families of light rays emanate from a region of infinite curvature. (timelike singularity)

Spacelike singularities are one of the features of uncharged non-rotating black holes, while timelike singularities are those that occur in exact solutions for charged or rotating black holes, where timelike or null geodesics can always avoid hitting the singularity.
What is referred to as a singularity does not belong to the classical spacetime. Where matter is compressed to such an extent that its effects on spacetime can no longer be described by general relativity, this is usually referred to as a "singularity". At such small scales and high densities, the relations between things must be described quantum-mechanically. If spacetime is shaped by the events that happen to objects, then when quantum objects influence the structure of spacetime it must be represented through a quantum theory. Since even in standard quantum theory time appears as a continuum variable, a radically new approach is required (Gambini & Pullin, 2011 , Oriti, 2009 & Rovelli, 2004)
It is expected that spacetime singularities will be cloaked by horizons. Although formation mechanisms for naked singularities have been proposed, the following conjecture is commonly regarded as valid:

• Cosmic Censorship Conjecture (Roger Penrose): Singularities are always hidden behind event horizons.

We emphasize that this conjecture is unproven in general relativity and thus does not have the strength of a theorem[4].

Singularity Theorems

Several singularity theorems can be proven from the purely geometrical properties of the spacetime model (Clarke, 1993). The most important is due to Hawking and Penrose (1970): Suppose (M, g_{ab}) is a time-orientable spacetime satisfying the following conditions:

1) $R_{ab}\, V^a\, V^b \geq 0$ for every non-spacelike V^a.
2) Generic condition such as chronology and geodesic completeness hold.
3) There are no closed timelike curves.
4) At least one of the following conditions holds:

• A compact achronal set without edge exists.
• A trapped surface exists.
• A p ∈ M exists such that the future (or past) expansion of the null geodesics through p becomes negative along every geodesic.

Then, (M, g_{ab}) must contain at least one incomplete timelike or null geodesic.

If this theorem is to be applied to the physical world, the assumption must be supported by empirical evidence. Condition 1 is satisfied if T^{ab} the energy-momentum tensor obeys the so-called strong energy condition: $T_{ab}V^aV^b \geq -\left(\frac{1}{2}\right) T_b^a$, for every timelike vector V^a if the energy-momentum is diagonal:

(Equation 36)

$$T_{\mu\mu} = (\rho, -P, -P, -P)$$

The strong energy condition can be written as ρ + 3P ≥ 0 and ρ + P ≥ 0. Condition 2 requires that every timelike or null geodesic experiences tidal forces at some point in its history. The first part of 4 requires that at least at one time, the Universe is closed and the spacelike slice corresponding to such a time is intersected more than

[4] The classic references on spacetime singularities are Hawking and Ellis (1973) and Clarke (1993).

once by an endless timelike curve. The trapped surfaces mentioned in the second part of 4 refer to horizons arising from gravitational collapse. The third part of 4 requires that the Universe is undergoing a re-collapse in the past or future.
The theorem is purely geometrical and does not invoke any physical laws. Theorems of this kind are consequences of gravitational focussing of congruences. A congruence is a family of curves such that precisely one and only one timelike geodesic curve passes through each point. For a time-orientable model if V^a is the timelike tangent vector to the congruence, we can write the spatial part of the metric tensor as:

(Equation 37)

$$h_{ab} = g_{ab} + V_a V_b.$$

For a given congruence of timelike geodesics we can define the expansion, shear and twist tensors as follows:

(Equation 38)

$$\theta_{ab} = V_{(i;l)} h^i_a h^l_b,$$

$$\sigma_{ab} = \theta_{ab} - \frac{1}{3} h_{ab}\theta,$$

$$\omega_{ab} = h^i_a h^l_b V_{[i;l]}.$$

Here the volume expansion θ is defined as:

(Equation 39)

$$\theta = \theta_{ab} h^{ab} = \nabla_a V^a = V^a{}_{;a}.$$

The rate of change of the volume expansion as one moves along the timelike geodesic curves in the congruence is given by the Raychaudhuri equation:

(Equation 40)

$$\frac{d\theta}{d\tau} = -R_{ab} V^a V^b - \frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab},$$

or
(Equation 41)

$$\frac{d\theta}{d\tau} = -R_{ab} V^a V^b - \frac{1}{3}\theta^2 - 2\sigma^2 + 2\omega^2.$$

Now we can use Einstein's field equations to relate the properties of the congruence with the spacetime curvature:

(Equation 42)

$$R_{ab} V^a V^b = \kappa \left[T_{ab} V^a V^b + \frac{1}{2} T \right].$$

The term $T_{ab}V^aV^b$ represents the energy density as measured by a timelike observer with unit tangent speed V^a. The weak energy condition then states that:

(Equation 43)

$$T_{ab}V^aV^b \geq 0. \quad \text{WEC}$$

A stronger condition is:

(Equation 44)

$$T_{ab}V^aV^b + \frac{1}{2}T \geq 0. \quad \text{SEC}$$

Note that this condition follows from the Raychaudhuri equation (3.40)

(Equation 45)

$$R_{ab}V^aV^b \geq 0.$$

We then see that the conditions of the Hawking-Penrose theorem imply focussing of the congruence takes place:

(Equation 46)

$$\frac{d\theta}{d\tau} \leq -\frac{\theta^2}{3},$$

where we have used that both shear and twist vanish. Equation 42 shows that the volume expansion of the congruence must necessarily decrease along the timelike geodesics. By integration we obtain:

(Equation 47)

$$\frac{1}{\theta} \geq \frac{1}{\theta_0} + \frac{\tau}{3},$$

where θ_0 is the initial value of the expansion. Thus, $\theta \to -\infty$ in a finite proper time $\tau \leq \frac{3}{|\theta_0|}$ means that whenever a focussing of timelike geodesic congruence occurs, a singularity must develop in the spacetime model. The non-spacelike geodesics become inextendible in such a situation.
A closely related theorem due to Hawking (1967) states:

Suppose (M, g_{ab}) is a time-orientable spacetime satisfying the following conditions:
1) $T_{ab}V^aV^b \geq 0$ for every non-spacelike V^a
2) A compact spacelike slice $\Sigma \subset M$ without edge exists.
3) The normal congruence to Σ is everywhere converging (or diverging).

Then (M, g_{ab}) is geodesically incomplete in the timelike sense.
Spacetime models can be classified according to the type of incompleteness they admit. A C^k inextendible spacetime model is one which does not possess C^k extensions of the manifold allowing the incomplete curves to be extended. The index k measures the degree of singularity of the spacetime model. The smaller k is, the stronger the singularity. (Clarke 1993)
The singularity theorems do not appear to apply to the Universe as a whole, since there is increasing evidence that the energy conditions are violated on large scales. (Riess et al. 1998; Perlmutter et al. 1999)

Conclusion

No light, not even X-rays, can escape from a black hole's trap. NASA's telescopes study black holes by looking around them and near the event horizon. The temperature of matter being dragged towards a black hole rises to millions of degrees, so it emits X-rays. Also, the warping of space around a black hole due to its immense gravity will be considerable. The effect of this invisible gravitational pull may be observed on stars and other objects. Hence, by studying the X-rays emitted from objects very close to the event horizon and the effect of the black hole's gravitational field on stars and other objects, its behavior is studied.

A stellar-mass black hole can form within seconds after the collapse of a very massive star. These relatively small black holes can also form from the merger of the dense remnants of two neutron stars. Black holes may also arise from the merger of a neutron star and a black hole, or the collision of two black holes with each other. The timescale for black hole formation by these mechanisms is rapid. However, the formation time for supermassive black holes may span billions of years, though their formation timescale is not definitively known.

Astronomers look at the motions of stars at the centers of galaxies to answer this question. These motions indicate the presence of an extremely massive object whose size is determined by the stars' velocities. Any matter swallowed by the black hole will add to its mass. A black hole's gravity will never dissipate from the universe. There is no way for an entire galaxy to be swallowed by a black hole. The gravitational field of the supermassive black holes residing at the centers of galaxies is immense, but not immense enough to swallow the entire galaxy.

The Sun will never turn into a black hole because its mass is insufficient for a large explosion. Instead, the Sun could turn into a dense object called a white dwarf. But if we hypothetically assume the Sun suddenly turned into a black hole, the orbits of the planets around it would not change due to gravity's uniformity.

As mentioned, stellar-mass black holes will form from the explosive demise of a very massive star. Elements like carbon, oxygen and nitrogen will be released as gas into space from such explosions. Similar elements will be dispersed around from the merger of two neutron stars, two black holes, or a neutron star and black hole. These elements are vital for sustaining life and may one day form part of new planets. The shockwaves from stellar explosions could potentially trigger the formation of new stars and solar systems. Thus, we owe our existence on Earth to the black hole-forming explosions from long ago.

The universe is a vast place. The region over which a black hole's gravitational field is appreciable is minuscule compared to the size of a galaxy. This holds true even for the supermassive black hole at the center of the Milky Way. Perhaps most stars formed near this black hole were swallowed by it, but more distant stars are safe from being devoured. Since the black hole's mass is a few million times the Sun's mass, devouring Sun-like stars will only increase its mass by a tiny amount. So, there is no danger to the Earth or the rest of the Milky Way.

Future galactic collisions will increase black hole sizes. However, these collisions will not occur indefinitely, as the universe is vast and expanding, so the likelihood of a runaway black hole effect will be minuscule. The event horizon is an imaginary surface that cloaks the black hole. Anything, even light and heat, passing through this surface has no return path to the outside. So, if heat cannot get out of the black hole's interior to the outside universe, the interior has no meaningful temperature for the exterior environment. Yet black holes do possess a temperature.

For the first time in 1972, a physicist named Jacob Bekenstein showed that the area of the event horizon represents the entropy of a black hole. Entropy is closely related to thermodynamics, so other physicists became perplexed. Hawking ended the confusion about black hole temperature by proposing his famous Hawking radiation theory.

Much research has been done on this topic. From data obtained across different wavelengths, particularly X-ray observations, collisions between black holes seem plausible. The gravitational forces in black hole collisions can accelerate matter to tremendously high speeds. When this compressed matter heats up, its temperature reaches millions of degrees. Therefore, most of its electromagnetic radiation is in the ultraviolet, X-ray, and even gamma-ray regimes.

As mentioned, black holes exist across a range of masses from stellar (a few times the Sun's mass) to supermassive (millions or billions of solar masses). Astronomers believe collisions between black holes of any mass are possible.

The collision of stellar-mass black holes results in a gamma-ray burst. The amount of energy released is enormous and will last no more than a few seconds.

When a massive star contracts under its own gravitational force, a black hole form. According to Einstein's theory, the star's matter is crushed to an infinitely small point of zero volume and infinite density called a singularity. In other words, the matter vanishes. Some scientists hypothesize the matter may re-emerge from an empty region called a white hole in another universe.

A black hole is composed of nothing but space and time itself. It can be thought of as a deep well in spacetime. At the very center, there may exist a superdense particle of matter.

Finally, the facts gleaned about black holes are:

- Black holes are not tunnels.
- Some black holes spin.

- Black holes are not always black.
- Black holes were not discovered by Einstein, but first by John Michell in 1783.
- Black holes are noisy.
- Anything can turn into a black hole.
- The laws of physics break down at the center of a black hole.

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