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Research Article

How The 3D Geometrical Patterns Can Be Used To Develop The Algebraic Thinking.

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ABSTRACT

This paper presents a teaching method that aims to restore the meaning and dynamism to the study of the remarkable transition from the geometric to the algebraic thinking in the learning of mathematics. The aim is to provide a geometric framework for the development of algebraic thinking. To this end, and in order to facilitate the links between theory and practice, our research is based on some activities that can be carried out in the classroom, assessment tools and student productions using didactic and IT tools. Our key idea it to ask a sample of middle-school scientific students to use some 3D shapes (volumes), and to try proposing some remarkable identities of order three, that generalize that of order two. This enables us to have some first answers to our main research problematic: "How the 3D geometry can be used to develop the algebraic thinking".

Keywords: Algebraic thinking, geometric thinking, Math. Education, ICT, didactic pattern, remarkable identities.

1 Introduction

In the framework that aims to provide an in-depth understanding of how geometric thinking evolves towards algebraic, highlighting pedagogical implications and teaching strategies to facilitate this transition. It integrates elements of empirical research, pedagogical theory and the analysis of innovative practices, as well as the analysis of innovative pedagogical practices and their impact on learning. It also examines mathematical models in which geometric thinking facilitates the understanding of algebraic concepts such as remarkable identity $(a + b)^3$,$(a - b)^3$, $a^3 - b^3$,and the analysis of students' production errors.
Geometric thinking is The geometric approach in mathematics focuses on shapes, sizes, spatial properties and dimensions. Another important aspect is to develop students' ability to generalize. In geometry, they often deal with specific cases, but in algebra, they need to learn to apply rules and formulas in a variety of contexts. Teachers can encourage this by presenting situations where an algebraic formula can be applied to several different geometric problems, reinforcing the understanding that algebra provides a general language for expressing mathematical relationships.
Geometry, through its objects, its statements, its methods and the representations it proposes, is involved in many branches of mathematics and science, sometimes in unexpected ways. The teaching of geometry is often defended on the grounds that it prepares students for mathematical reasoning, i.e., a blend of deductive reasoning and inductive imagination, activated by our familiar manipulation of images. There's no point in repeating here all the arguments that have been put forward in defense of geometry teaching, nor indeed in refuting them (Brousseau, 2010).

Geometry as a model of sensible space, "a *mode of knowledge production relating to the material objects of space*" (Perrin-Glorian & Salin, 2010), which provides the means for predicting and controlling the problems that arise in the surrounding space. Bkouche (1990) speaks of "the *science of spatial situations*", Chevallard and Jullien (1990-91) consider geometry as "the *technology of space, the theory of the practical mastery of space*" and write "*geometry starts from the sensible world to constitute it into a geometric world, that of points, straight lines, circles, spheres, curves, surfaces and volumes, etc.*" (Houdement, 2021).
Geometry offers teachers an opportunity to provoke in their students an activity recognized as authentically mathematical by most mathematicians themselves. This is not the case with elementary arithmetic, which is absorbed into algebra, or with algebra itself, which is often equated with calculus and even reduced to

algorithms, not to mention statistics, whose content is scarcely recognized as mathematical. This is partly due to the fact that geometry offers a whole felt of interrelated statements about a small number of objects, which can be approached through a tightly woven web of theorems, and partly to the age and luxuriant profusion of approaches or points of view on the subject (Brousseau, 2010).

However, teaching geometry for all necessarily has at least two aims: the teaching of geometric knowledge, i.e. a coherent theoretical framework governed by axiomatic (explicit or otherwise), and the use of this framework to solve concrete problems. There is also a third: geometry as a means of representation for other fields of knowledge, including within mathematics itself, what is sometimes called geometrical thinking or geometrical intuition, constituting a powerful heuristic tool by virtue of the fact that we can transfer intuitions arising from our relationship with space into these fields (Douady, 1994). The Kahane Commission report (2002) gives all these objectives to the teaching of geometry, adding to them the learning of reasoning which, for us, does not boil down to the learning of demonstration. This is not the aspect we emphasize in this article, but it is of course very important to us too (Glorian, 2022).

2 Theoretical framework:

At the operative level, algebraic thinking is deployed by means of:
1. A set of special reasoning, such as generalizing, reasoning analytically, symbolizing and operating on symbols; reasoning about relationships between variables, especially functional relationships; reasoning in terms of structures, etc.
2. Ways of approaching concepts involved in algebraic activities, such as treating equality as an equivalence relation; leaving operations in abeyance; seeing a numerical expression as an object in itself and not just as a chain of calculations, etc., and modes of representation and ways of operating on these representations.

In this conceptualization, the algebraic character of mathematical thought or activity does not lie in the nature of the ostensives, in the sense of Bosch and Chevallard (1999), i.e. in the presence of alphanumeric signs. Rather, it lies in the nature of the non-ostensives, i.e., in the meanings of the concepts and the nature of the reasoning involved. In this sense, and more precisely according to Radford (2018), algebraic thinking resorts to analytically processed indeterminate quantities and to idiosyncratic or specific, culturally and historically evolved modes of representing / symbolizing these indeterminate quantities and their operations.
Since analytic consists in treating known and unknown data together on the basis of properties, it is characteristic of algebraic reasoning. Analytic is not involved in arithmetic, in particular, as Radford points out for trial-and-error strategies: "*trial-and-error methods fail to satisfy the condition of analytic*" (Radford, 2014, p. 260). Situations involving the generalization of geometric patterns play a central role in the work of the EARLY ALGEBRA movement. Their use in research has shown that young students can implement analytic and thus develop algebraic thinking before encountering the formal symbolism introduced in high school. We see analytic as a technological ingredient in the numerical-algebraic activity developed in particular in praxeologies involving generalization-type problems.

In mathematics, algebra and geometry make cultural use of semiotic registers (Duval, 1991, 2006) such as graphic representations, geometric drawings, algebraic expressions and so on. However, students can also create their own personal semiotic representations, which will have to evolve towards the representations fixed in mathematical knowledge. In the development of algebraic thinking, the tasks of processing and converting between different registers, whether spontaneous or not, are very important. In this respect, natural language is an essential register for giving meaning to the reasoning developed by students. We follow the analytical framework of Squalli et al. (2020) to categorize reasoning along two dimensions: analytic and the nature of semiotic registers. Three main categories of reasoning emerge from this categorization.

- **Non-analytical reasoning**: This category includes reasoning that is characteristic of an arithmetical approach to solving problems. To determine the values of the unknowns, the student operates on known data and relationships. At no point does he operate on an unknown or on a non-determined number (for example, a variable or a parameter). This type of reasoning is highly effective in solving connected problems.
- **Analytical reasoning**: This is reasoning that respects all the characteristics of analytical reasoning as defined above. In this type of reasoning, the student considers the unknown, represents it by a symbol, uses this representation to express the relationships between the known data and the other unknowns of the problem, and operates on these representations to form the equation and find the values of the unknowns.
- **Reasoning with analytical tendencies**: We include three classes of reasoning in this category. The first is hypothetico-deductive reasoning, in which the student assigns a given value to an unknown knowing it to be false, acts as if this unknown possessed this value, operates on the relations and generates the values of the other unknowns. He then reasons on the relations and values produced to find the exact value of the initial unknown. False-position reasoning is an example of such reasoning. In this type of reasoning, the subject acts as if the value of the unknown were known, but instead of operating on a representation of this unknown, he operates on a false but determined value. For this reason, we consider this type of reasoning

to have an analytical tendency, but not to be analytical. The second class includes reasoning in which the student considers the unknowns momentarily as variables. To find the values of these variables that respect the conditions of the problem, he does not operate on them - as in the case of analytical reasoning - but on their numerical instantiations. This is the case with functional reasoning, a prototypical example of which is presented below. The third class includes reasoning in which the student considers the unknown, represents it explicitly, uses this representation to translate the relations between the unknown and the known, but does not operate on these representations to find the values of the unknowns. It is for this last reason that the degree of analytic of the reasoning is not considered optimal. (Squalli et al., 2020).

For each major category of reasoning, sub-categories related to each category are used. These are specific types of reasoning that can be used to find possible answers to the problem.

The concept of didactic transposition (Chevallard 1985) was taken up by Balacheff (1994) and reworked, taking into account the constraints associated with learning knowledge in a computer environment, under the name of computer transposition. The knowledge taught in a conventional teaching situation is different from the knowledge taught with a computer, which can be summarized as follows

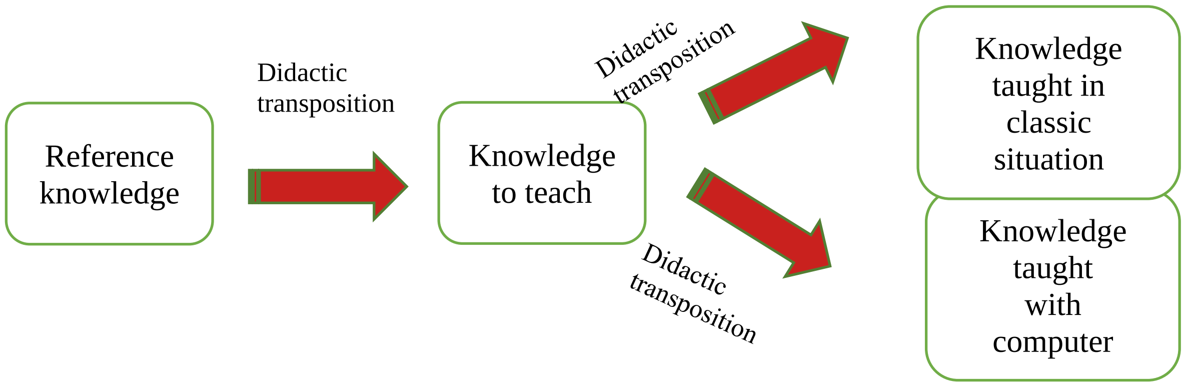


Figure 1. Didactic triangle and computer.

Balacheff (1994) explains that the constraints of didactic transposition are compounded, or rather combined, with those of computer modeling and implementation. This researcher defines two types of constraints linked to computer transposition: computable modeling constraints, and software and hardware constraints for computer media. The former relate to the representation and internal processing of knowledge in the machine, and the latter to representation and processing at the interface level, in other words, what is "visible" to the subject. Giving the example of Geogebra software, which has an internal representation of geometric objects derived from analytic geometry on a real number model, and an interface offering a representation of these objects in the form of a finite paving of pixels. He points out that these representations are not transparent: "since representation systems have their own characteristics, the internal universe and the interface combine generative effects and phenomena that are not intrinsic to the entities represented." (ibid., p.16).

3 Research subject.

The transition from the algebraic thinking to the geometric one is a key step in many areas of mathematics, as linear algebra, analytic geometry or differential geometry. This transition enables the learner to go from a representation based on geometric objects such as points, lines, surfaces or volumes to another one based on equations and algebraic expressions. The transition between the algebraic and the geometric thinking is a fascinating exploration of the relationship between algebraic expressions and geometric forms.

A notable reference in this field is (Strang, 2016), where the author explores in detail the relationship between algebraic and geometric concepts in linear algebra, focusing on the geometric visualization of operations on vectors and matrices. This approach helps the students to intuitively understand abstract concepts such as linear independence, linear transformations and vector spaces.

For example, in analytical geometry, the transition from the algebraic to the geometric is often illustrated by the graphical representation of equations : the equation of a straight line “y = mx + b” can be visualized as a straight line with a slope “m” and a y-intercept “b”. Similarly, quadratic and cubic equations can be represented by geometric curves (Strang, 2016). While, in differential geometry, the transition between the algebraic and the geometric is crucial to understanding concepts such as curves and differentiable surfaces. These geometric objects are often described using algebraic equations involving partial derivatives, but they can also be apprehended geometrically by considering their tangents, curvatures and other properties (Strang, 2016).

The geometric thinking is an indispensable tool to develop certain facets of the algebraic thinking, and vice versa. They should not be considered in a hierarchical manner, but rather as evolving simultaneously over the course of learning. In this way, the reasoning, concepts and modes of representation that are typically learned in algebra or when working with geometric shapes enrich the development of both types of thinking, and are indispensable tools for their co-development. The evolution of these two types of thinking and the work done non-exclusively in each of the activities also constitute an essential overlap for the development of geometric and algebraic thinking. At the operative level, we consider algebraic thinking in three dimensions, i.e. as :

1. A set of special reasoning;
2. A special relationship with the concepts involved in the activities;
3. A way of communicating.

These three distinct components are not hierarchically placed, and must be seen as constantly interrelated. Let us take as an example the realization of a geometric modeling activity that involves geometric representation through the study of algebraic curves. the remarkable identity "$(a + b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$ " can be visualized geometrically by considering perfect cubes of side lengths, and other powers can be interpreted geometrically in terms of properties of geometric shapes such as rectangles, squares, parallelized rectangles (Robert, 2018).

This way of the algebra "looking" is not in line with the research focused on algebraic thinking (Kieran 2007): the definition of this thinking is not directly linked to the ability to use mathematical writing involving letters. On the contrary, it has more to do with a particular way of reasoning about problems, which can be developed in a numerical framework, well before the first learning in secondary school (Kieran 2007; Cai & Knuth 2005 & 2011; Radford 2006; 2008 & 2014; Vlassis, 2015).

Indeed, remarkable identity activities in geometry encourage students to be creative, change registers and look for regularities. They help develop students' algebraic thinking, which for us is associated with reasoning, argumentation, the idea of taking an interest in the structure of figures, mathematical relationships between remarkable identities in figures rather than calculations with numbers, working with volumes and extracting relationships that lead us to our "remarkable identity" objective, and generalizing. We believe that algebraic thinking develops if students are encouraged to look for regularities, to formulate conjectures that they will try to invalidate or validate (the proof is sometimes not attainable in the early middle school year to arrange volumes in order to have a relationship adapt to the remarkable identity). These practices of teaching geometry on the grounds that it would prepare pupils for mathematical reasoning, i.e. a blend of deductive reasoning and inductive imagination, activated by our familiar manipulation of images. There's no point in repeating here all the arguments that have been put forward in defense of geometry teaching, nor indeed in refuting them.*"Being a geometer means not confusing evidence derived from intuition with experimental information, or the result of an experiment with the conclusion of a line of reasoning. It means deciding, in principle, what credit to give to each of these functions of thought"* (Gonseth,1945).

As a natural continuation of our last work (Bentaher & all. 2023), we continue our reflection on how pupils use algebraic generalization and symbolization during the transition from a geometric thinking process to the algebraic one. In the last work, we focused on remarkable identities of order two: $(a+b)^2$, $(a-b)^2$ and a^2-b^2. We asked a sample of 10-12 years old students from middle school to draw some geometric figures of squares and rectangles and asked them to conjecture the three well-known remarkable identities, $(a+b)^2$, $(a-b)^2$ and a^2-b^2. We were impressed by the many difficulties met by this students to this end. In this work, we would like to go through remarkable identities of order two: $(a+b)^3$, $(a-b)^3$ and a^3-b^3. In fact, it will be interesting to discover how the same students will behave in a similar situation, but with a high level of difficulty? How they will behave in the transition from equation of second degree to that of three? How they will behave in the transition from 2D patterns to that in 3D-dimension?

With a view to the transition from geometric to algebraic thinking, we may ask many other questions, for example :

- what kind of geometric reasoning is used by students in all three cycles of junior high school?
- Can middle school students develop geometric thinking?
- What mistakes are made in students' productions in the transition from geometric to algebraic thinking?
- What role does ICT play in the development of both algebraic and geometric thinking?
- We analyze the productions by level of error used by students from the perspective that geometric thinking articulates the geometric and algebraic dimensions.

4 The experimentation: the progress and its analysis.

4.1 Global presentation.

For our purpose mentioned here above, the experimentation was leaded with a sample of 150 students of 12-14 years from a Moroccan middle school (Imam Malik in Taza): 50 from them are from the first year of the middle school level, 50 from the second one and 50 from the third one. Each one of these three groups is for its part divided randomly into two groups: an observation group of 25 students and an experimental group of 25 students. Thus we have:

- A total of 150 students from middle school of 12-14 years;
- 50 students form each one of the scholarship levels: 50 from the first year, 50 from the second and 50 from the third of the middle school;

- Two groups of 75 students each one: an observation group and an experimental group;
- Each one of this 75 students is divided in three subgroups following the scholarship level of their members;
- Finally we got 6 subgroups of 25 members: three of them are from the observation group, and three of them are from are from the experimental one.

	Observation Group	Experimental Group	Total
First year level in middle school	25 persons	25 persons	50 persons
Second year level in middle school	25 persons	25 persons	50 persons
Third year level in middle school	25 persons	25 persons	50 persons
Total	75 persons	75 persons	150 persons

Table 1. Table that summarizes the partition of the 150 students.

Four activities, that will be more detailed later, were proposed to this 150 students under the supervision of a mathematics teachers, and with their approbations, that of their teachers and that of the administration school, this activities were filmed, in order to be later analyzed.
One may ask why we divide the full sample in two subgroups : an observation one and another experimental one. The goal is to propose the same activities to the two groups, the only difference is that the experimental groups will use GeoGebra as an interactive geometry and calculus free software which is designed for learning these mathematics in a school setting, ranging from primary to university level. We will compare the averages of the marks obtained by each subgroups in the first and last activity. This may enable us to have some answers to one of our problematic questions, which is : What role does ICT play in the development of both algebraic and geometric thinking?

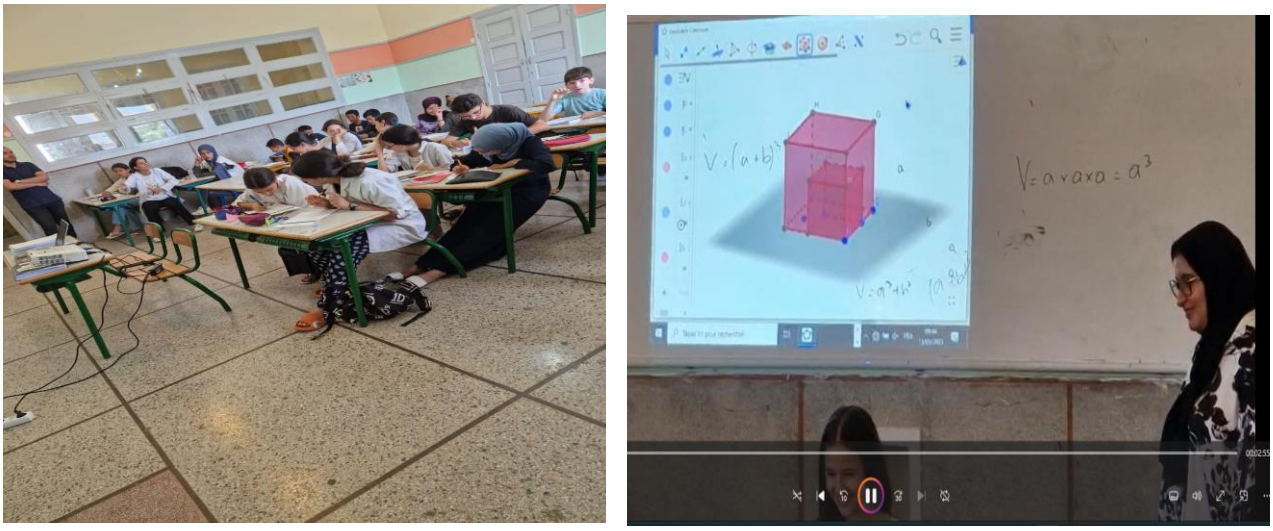


Figure 2. Some screenshots of the proposed activities

4.2 The first activity.

The first activity was in the form of a diagnostic test on the students' ability to manipulate geometric patterns in calculating areas and volumes. To do this end, we proposed to the 150 students 4 exercises:

- In the first exercise (see Figure 3), the students were asked to compute the areas of the patterns (1) and (2), and then deduce that of the pattern (3). The aim is to verify how (wrong or false) the students use the juxtaposition of motifs as an addition of numbers, in otherwise how they use their geometric thinking to develop their algebraic one;
- In the second exercise (see Figure 4), the order was the same than that of the first exercise the students were asked to compute the areas of the patterns (1) and (2), and then deduce that of the pattern (3). The aim here is slightly different, we aim to verify how (wrong or false) the students use the omission of motifs as a subtraction of numbers;
- In the third exercise (see Figure 5), we proposed 3 shapes to the all of 150 students, and then asked them to calculate for each pattern, the area without any additional indication. The aim here is to verify how (wrong or false) the students use their geometric thinking to develop their algebraic one;

- In the fourth exercise (see Figure 6), we asked to the all of 150 students, to compute the volumes of the patterns (1) and (2), and then deduce that of the pattern (3). The aim here is to verify how (wrong or false) the students use their 2D geometric thinking to develop their 3D one.

After correcting the students answers (see some screenshots in Figure 7), our remarks about the most frequent students errors or difficulties are the following:
• Many confusion between the terms surfaces and perimeters;
• Some visual or strategy difficulties during the partition of a pattern (area or volumes);
• Some lack of reasoning during the summation of calculated areas or volumes;
• Some errors during the conversion of the unity;
• Many conceptualization difficulties during the treatment of a volume of a figure in a patterns.

We correct, with the assistance of their teachers, the production of each one of this 150 students sample and compute the average of each subgroup (see Table 2).

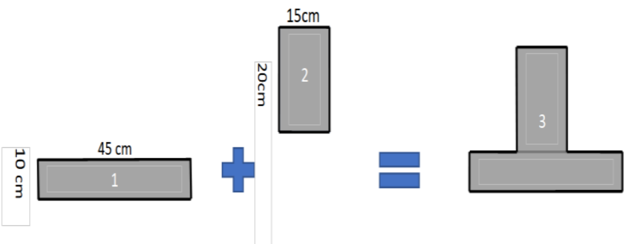


Figure 3. Adding areas by juxtaposing motifs.

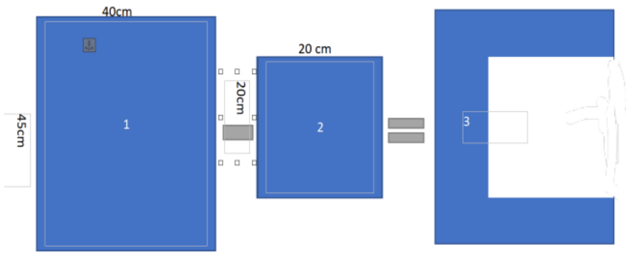


Figure 4. Substracting areas by omitting motifs.

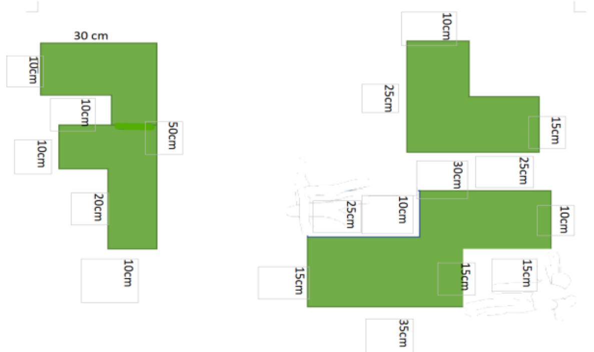


Figure 5. Computing areas without any additional indication

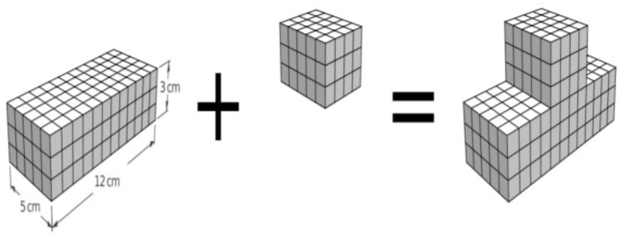


Figure 6. Computing volumes by juxtaposing patterns

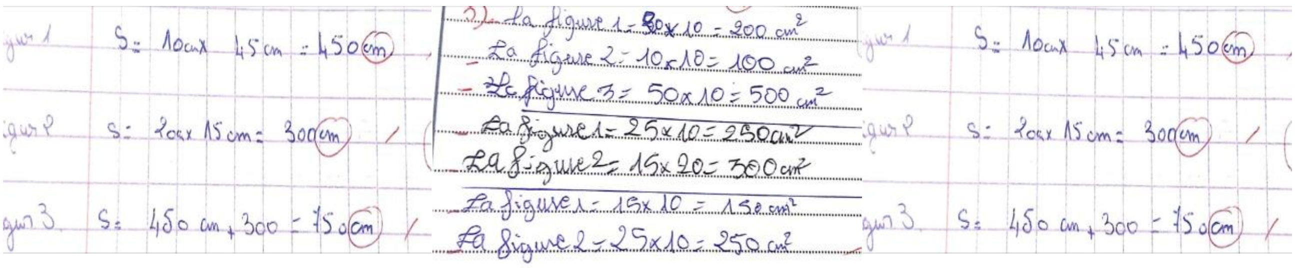


Figure 7. Some screenshots of the errors made by the students while answering the exercises proposed in the first activity.

	Experimental Group	Group Observation	Global Average
First year level in middle school	9,01	8,82	8,91
Second year level in middle school	6,27	10,58	8,42
Third year level in middle school	13,43	14	13,71
Global Average	9,57	11,13	10,35

Table 2. Table that summarizes the different averages of marks of the 6 subgroups after the first activity.

The first lecture of this data (collected in the first activity of the whole experimentation) enable us to remark the following:

- For a fixed level (expected the second one), the difference between the averages of the observation group and that of the experimental one is almost null;
- If one wish to compare the full observation group with the full experimental one, the data says that apparently, however randomly divided, the experimental group is better than the observation one in the computation of areas and volumes by using patterns;
- If one wish to compare levels, the data says that apparently, the computation of areas and volumes skills of the students from the first level and second one are almost the same, but those of the third year level are so much better.

4.3 The second activity.

After a brief recalling on the three remarkable identities of the second degree $(a+b)^{2,}$ $(a-b)^2$ and a^2-b^2, we propose to the all 150 students 3 exercices:

- Firstly we asked to propose their intuitive formula for the first remarkable identity of degree three, namely $(a+b)^3$. We were really surprised by their spontaneous answers (see some screenshots in Figure 8.). All the 150 students gave the same answer.

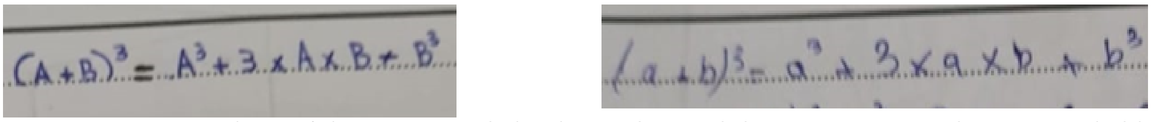


Figure 8. Some screenshots of the errors made by the students while trying to generalize a remarkable identity from degree two to degree three.

- Secondly, we asked them to find the true formula, by using this indication:$(a+b)^3=(a+b)^2(a+b)$. Someones succeed, others fail. However, we are surprised by some answers, which we did not expect (see some screenshots in Figure 9.).

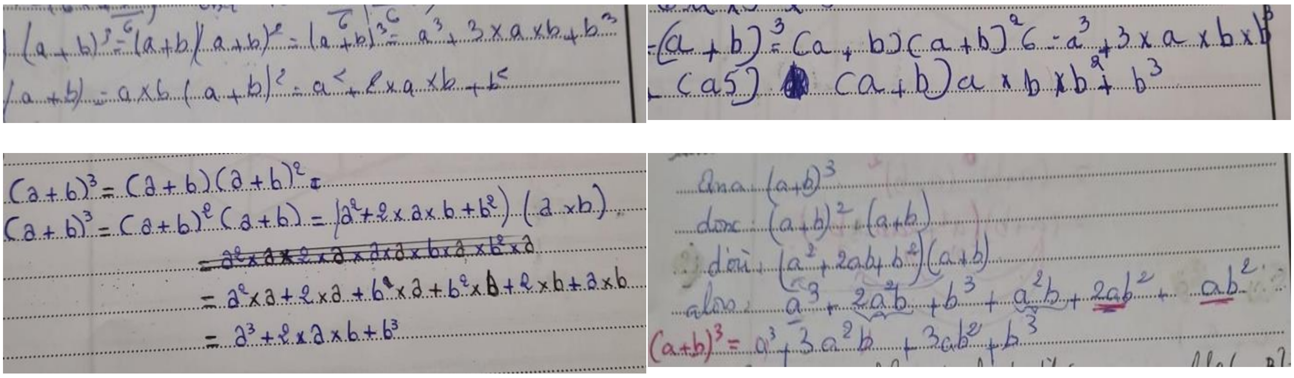


Figure 9. Some screenshots of the production of the students while trying to find (with an indication) a remarkable identity of degree three.

- Finally, we gave them the correct formula $(a+b)^3=a^3+3a^2b+3ab^2+b^3$ (after justifying it in the whiteboard). Then we ask them to apply it in the following literal cases: $(x+6)^3$, $(3x+1)^3$, $(2x+2)^3$, $(6x+5/6)^3$. Someones continue to use their wrong (however intuitive) formula, despite the teacher correcting it with them. Others write some unexpected and incomprehensible answers. Finally, others succeed to apply the correct formula (see some screenshots in Figure 10.).

To close our discussion of this second activity scheduled in the experimentation, we would share here below our analysis of the errors produced by the students, either when generalizing a second degree equation to a third one, or when following indication to find a correct equation, or during the simple and literal application of this equation:

- Forgetting terms: Some students shade out certain terms when developing the expression, neglecting elements like $3a^2b$ or $3ab^2$.
- Improper distribution: Some errors have occurred by incorrectly distributing powers, resulting in incorrect terms in the final result.
- Sign confusion: Students made errors while manipulating the signs in the expanded expression, leading to inconsistent results.

- Confusion with other remarkable identities: Some have mixed the formulas of remarkable identities, for example, confusing $(a + b)^3$ with $(a + b)^2$ or other expressions.
- Lack of simplification: Some students do not correctly simplify the expanded expression, leaving common terms uncombined.
- Factoring Problems: Some students have difficulties to correctly factor a cubic expression. This leads to errors in the opposite direction of development.
- Confusion of powers: Some errors can occur if students confuse powers of terms, such as cubing (a + b) instead of each individual term.
- Lack of justification: Some students may not properly explain each stage of their development, which can make it difficult for the teacher to understand the logic behind their.

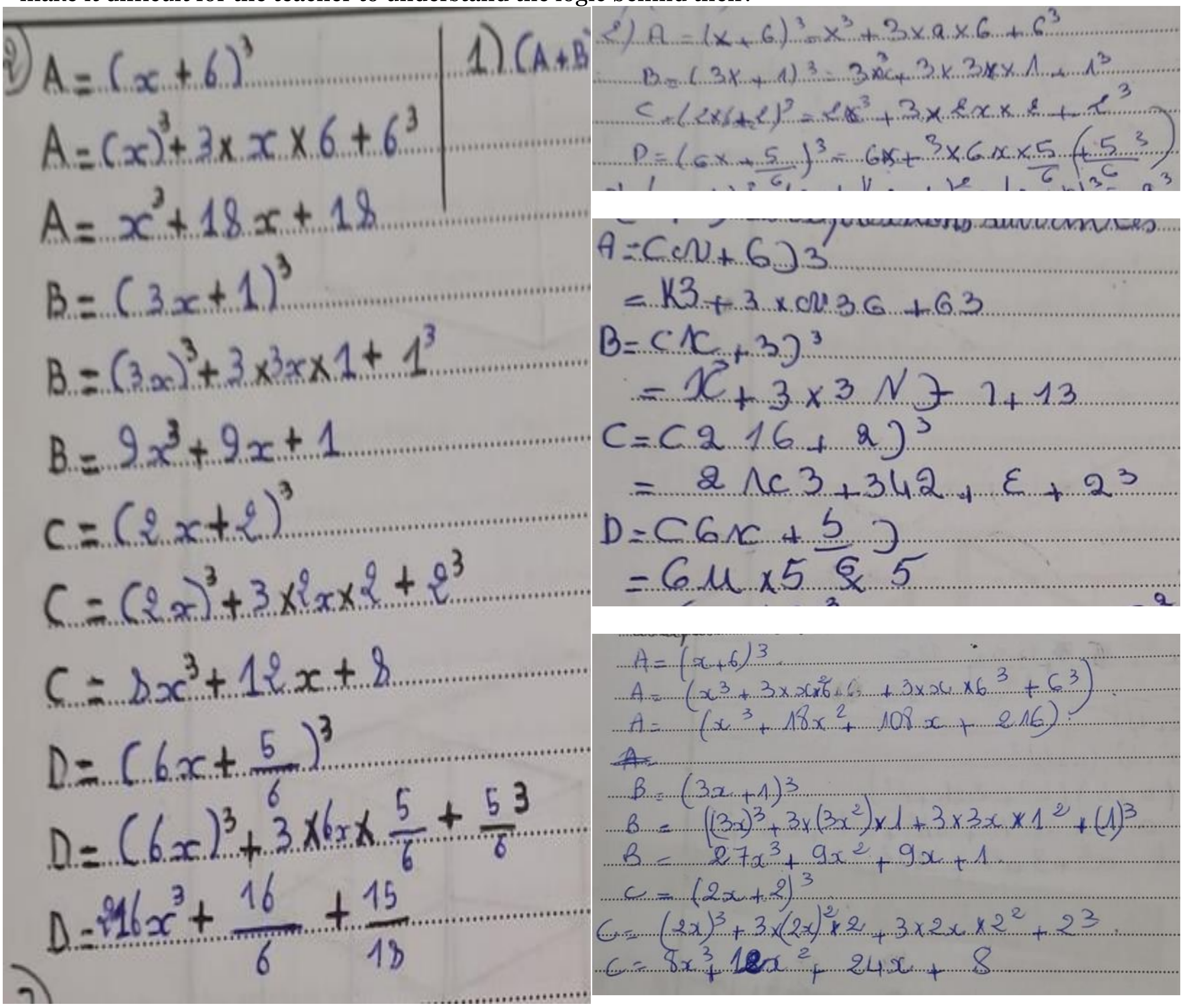


Figure 10. Some screenshots of the production of the students while trying to apply literally a remarkable identity of degree three.

4.4 The third activity.

The main goal of this third activity is to correct, by using a geometrical visualization, some erroneous representations of the students on the third remarkable identity of degree 3: $(a + b)^3$. In this activity we focused particularly on correcting this false representation: $(a+b)^3=a^3+b^3$. Here the first member of the equality must be interpreted as the volume of a cube of edge (a+b), while the second must be interpreted as the juxtaposition of two small cubes, one of edge a and the other of edge b. The instructions for the students were as follows:

- Draw a cube of edge (a+b), what is its volume;
- Draw and juxtapose two cubes, one on edge a and the other of edge b. What is the volume of the resulting shape;
- Justify by using only your geometrical visualization that the formula $(a+b)^3=a^3+b^3$ cannot be correct.

It is worth to note that although this activity was proposed to all 150 students, those of the observation group were asked to answer by drawing the shapes on a paper, while those of the experimental group were expected to answer by using a drawing software, GeoGebra. It is also worth to point out that before participating in this activity, the 75 students from the experimental group received a training to this free software provided by their teachers.

At the end of this activity, the students of the experimental group (GeoGebra group) succeeded in drawing some beautiful figures (see Figure 11.). from which they were able to geometrically visualize that the formula $(a+b)^3=a^3+b^3$ cannot be correct. They justify this that the juxtaposed volume of the two small cubes seems smaller than that of the large cube, because on should have to add more volume to the second shape to obtain the first one.

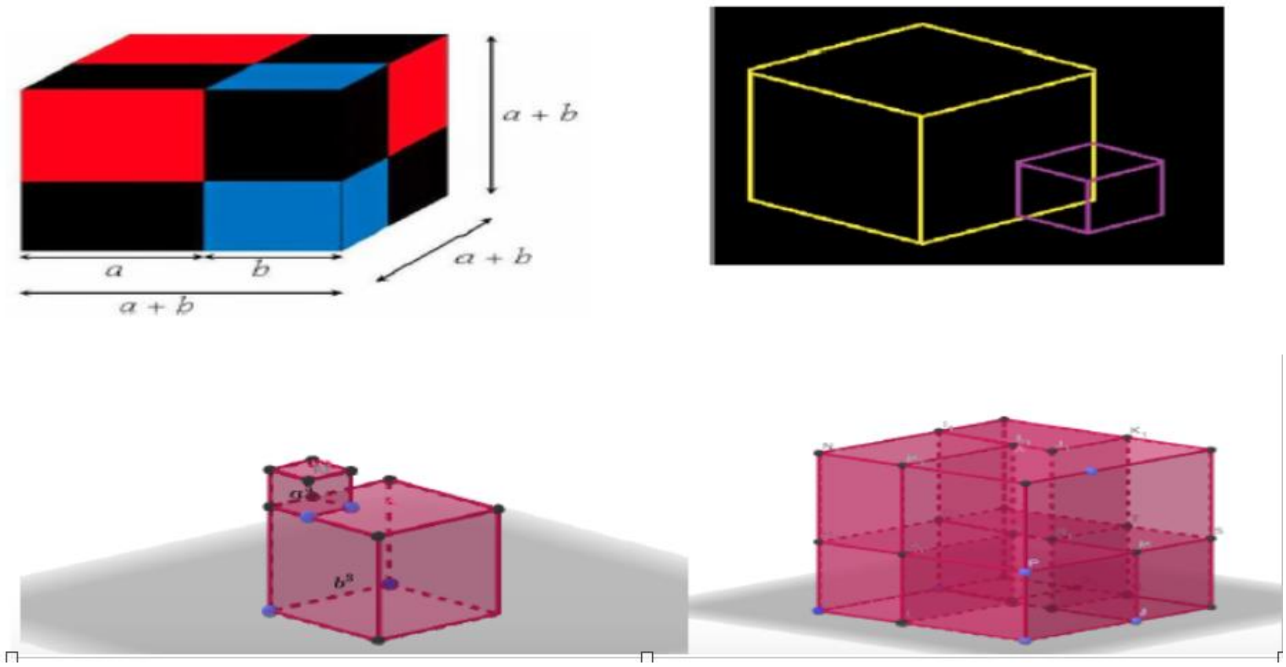


Figure 11. Some screenshots of shape drawn by the students from the experimental group to justify that (a+b)3 ≠ a3+b3,

Our lecture and analysis of the answers of the student from the experimental group led us to the following conclusion. When using GeoGebra to visualize geometrically in three-dimension, some algebraic identities, some errors can also occur. For example:

- Improper placement or positioning of objects in 3D space can result in incorrect representation of geometry;
- Inconsistencies in object properties: Not respecting the geometric properties of objects, such as angles or lengths, can lead to inconsistent models;
- Confusion between tools: GeoGebra offers various tools for geometry;
- Confusing the use of these tools can lead to errors in drawing construction;
- Perspective issues: Although GeoGebra is capable of handling 3D representation, an inadequate understanding of perspective can still lead to distorted representations;
- Excessive complexity: Adding too many elements or details can make the drawing confusing. It is essential to maintain clarity to facilitate understanding.

On the other hands, the students in the observation group (Draw in paper group) got lost in many incomprehensible, but also amusing figures (see Figure 12.). They did not have enough time to discuss the asked question.

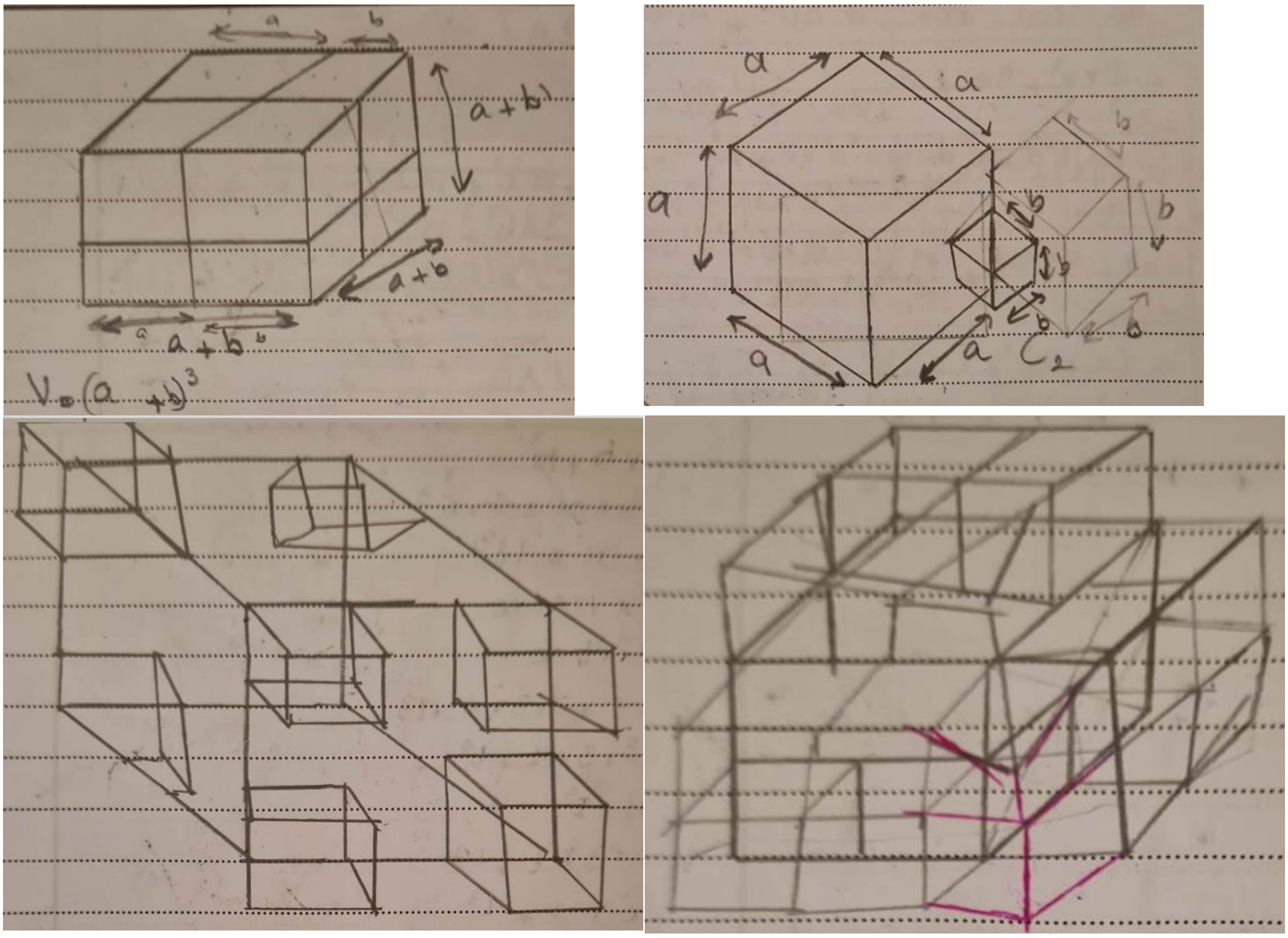


Figure 12. Some screenshots of shape drawn by the students from the observation group to justify that $(a+b)^3 \neq a^3+b^3$,

Our lecture and analysis of the answers of the student from the observation group led us to the following conclusion:

- Incorrect Perspective: Forgetting to depict perspective correctly can lead to inaccurate proportions and unrealistic drawings;
- Misjudged angles: Failing to correctly measure angles in three-dimensional space can result in inaccurate geometric representations;
- Confusion between projections: Students sometimes confuse different projections (like orthogonal projections) and not apply them correctly;
- Symmetry Issues: Failing to maintain correct symmetry in a three-dimensional object can lead to distorted designs;
- Confusion between line types: Failure to correctly use different line types to represent elements such as hidden edges, visible edges, etc., can result in incorrect interpretation of the drawing;
- Lack of precision: Lack of care and precision can lead to errors in execution, even if the conceptual understanding is correct. Encouraging students to be meticulous in their drawings can improve overall quality.

4.5 The fourth activity.

Our main purpose in this activity was to evaluate how the students use their geometrical visualization to compare two algebraic expressions. In this activity we focused particularly on comparing this two algebraic expressions: $(a-b)^3$ and a^3-b^3. Here the first expression should be interpreted as the volume of a cube of edge (a-b), with a>b. The second one should be interpreted as the omission of a small cube of edge b from a big one of edge b. The instructions for the students were the following:

- Draw a cube of edge (a-b), with a>b. What is its volume;
- Draw a big cube of edge a, and omit from this cube a small cube of edge b. What is the volume of the resulting shape;
- Justify by using only your geometrical visualization that the two expressions $(a-b)^3$ and a^3-b^3 cannot be equal;
- Which one of this expressions $(a-b)^3$ or $a^3- b^3$ is bigger.

As in the third activity, this fourth one was proposed to all 150 students, those of the observation group were asked to answer by drawing the shapes on a paper, while those of the experimental group were expected to answer by using the drawing software, GeoGebra. And, as expected, the students of the experimental group (the GeoGebra group) succeeded in drawing some beautiful figures (see Figure 13.) and in correctly answering the two last questions of the activity. On the other hands, the students in the observation group (Draw in paper group) got lost in many incomprehensible, but also amusing figures (see Figure 14.). They did not have enough time to discuss the asked question.

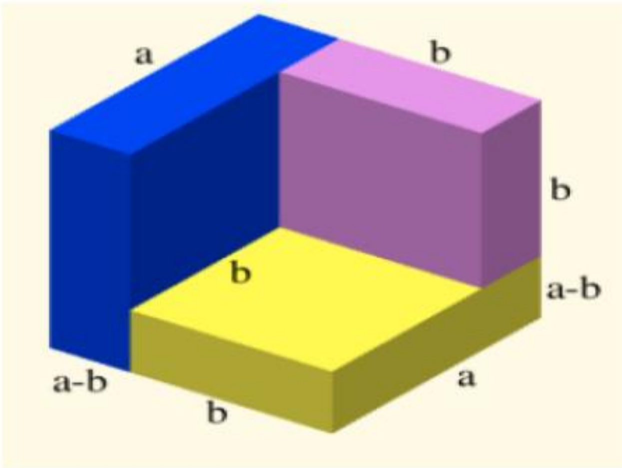
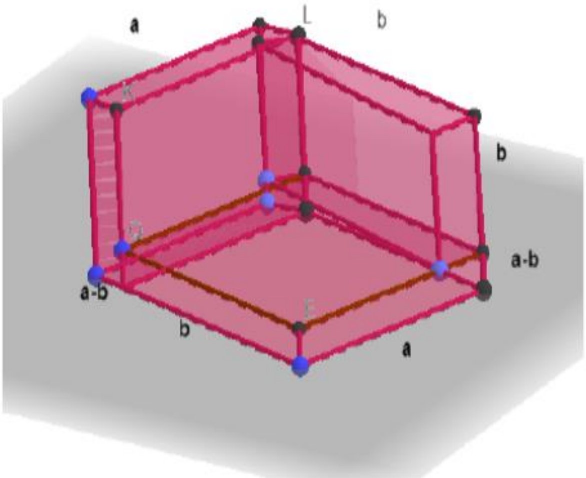


Figure 13. Some screenshots of shape drawn by the students from the experimental group (GeoGebra group) to compare the expressions $(a-b)^3$ and a^3-b^3.



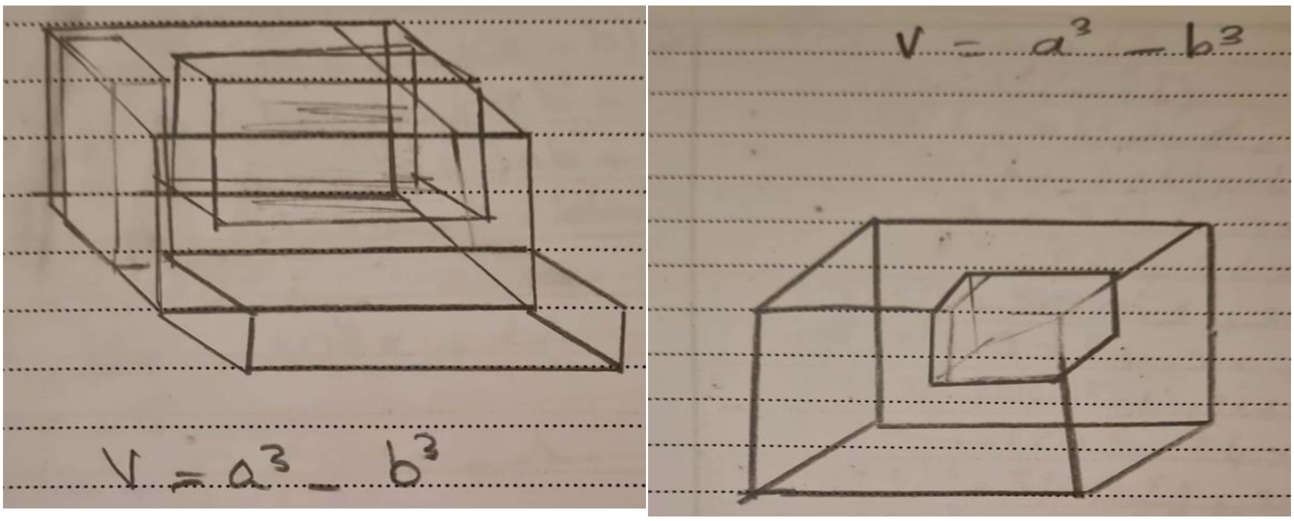


Figure 14. Some screenshots of shape drawn by the students from the observation group (Draw in paper group) to compare the expressions $(a-b)^3$ and a^3-b^3.

Let us recall that all the 150 students were evaluate at the beginning of the experimentation in order to have an idea about their background and skills to use their geometrical 2D-visualization to compare two polynomials expressions of degree two. Thus, We then judge it will be useful to evaluate them once again during the fourth activity. Table 3. summarize the progression of the averages of the students marks for all the subgroups. This will enable us to have an idea about their background and skills to use their geometrical 3D-visualization to compare two polynomials expressions of degree three.

	Experimental Group (GeoGebra Group)	Group Observation (Draw in Paper Group)	Global Average
First year level	9,01→ 7,79 (-1,12)	8,82 → 7,71 (-1,11)	8,91 → 7,75 (-1,16)
Second year level	6,27 → 5,34 (-0,93)	10,58 → 10,20 (-0,38)	8,42 → 7,77 (-0,65)
Third year level	13,43 → 11,58 (-1,85)	14 → 12,03 (-1,97)	13,71 → 11,8 (-1,91)
Global Average	9,57 → 8,23 (-1,4)	11,13 → 9,98 (-1,15)	10,35 → 9,1 (-1,24)

Table 3. Table that summarizes the different averages of marks of the 6 subsgroups after the fourth activity.

The first lecture of this table shows us a negative evolution for all the subgroups without any exception. Meaning that one may says that the students are better in the geometrical 2D-visualization than that of the geometrical 2D-visualization, especially in their final and third year in middle school. If one compare (-1,4) the evolution of the experimental Group (GeoGebra Group), with (-1,15) the evolution of the observation Group (Draw in Paper Group), one obviously that there is no significant difference. This leads us to answer once the classical question: How significantly do the ITC contribute to improve the mathematics learning?

Conclusion:

Overall, the student must work in physical space, in abstract space and create bridges between these two spaces in order to achieve their learning. Since this thought takes several years to develop, students are sometimes located in abstract space and sometimes in physical space. Moreover, this passage is far from smooth for the students. “Among all the areas of knowledge that students must enter, geometry is the one that requires the most complete cognitive activity, since it requires gesture, language and the look, there, it is necessary to construct, reason and see, inseparably. » (Duval, 2005).
This global vision of learning can represent a guide for analyzing, and even developing, the possible activities to present to students or even for analyzing and situating students' learning according to their productions and their verbalization. In this sense, the didactic reflection explored in this article concerns the spatial sense which is both distinct and inseparable from geometric and metric thinking. It does not directly deal with geometric objects (points, lines, figures or solids), but rather allows them to be treated in space. It is a way of thinking that allows students to represent and manipulate geometric objects physically and mentally, possibly preserving their spatial, geometric and metric properties. From this thinking, students can access mathematical reasoning such as deduction (which development represents that of cube?) or generalization (what are all the developments of cube?) and even refine their conceptualization using the elaboration of mental images that go beyond prototypical figures
During theses activities, skillfully orchestrated by the teacher, and by analyzing the patterns in the three remarkable identities, specifically for $(a + b)^3$, $(a - b)^3$ and $a^3 - b^3$, students can develop a deeper geometric understanding:

- Cubic pattern: Identify the cubic pattern in the identity $(a + b)^3$. It suggests a relationship with other parallelized rectangles whose side is the sum of “a” and “b”;
- Visual diagrams: Encourage students to create visual diagrams illustrating these patterns, highlighting the geometric structure inherent in the identity.
- Extension to other identities: Apply this understanding of patterns to other notable identities, such as $(a - b)^3$ or $a^3 - b^3$, by identifying the square patterns and products needed (see Figure 15).

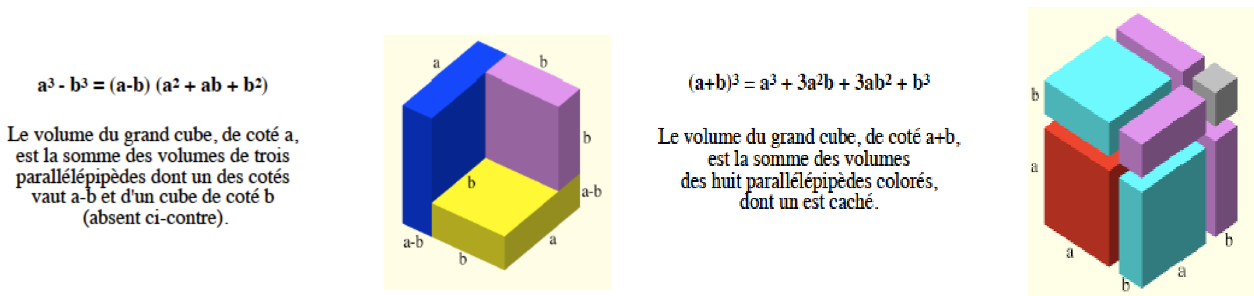


Figure 13. Some screenshots of shape to identifying the square patterns and products needed.

By analyzing patterns geometrically, students can better grasp how notable identities relate to specific geometric shapes, strengthening their overall understanding of mathematical concepts. To understand how algebra teaching can build on students' experience with algebra, much research has been conducted on the transition from geometry to algebra. But the transition for students from a geometric mode of thinking to an algebraic mode of thinking is far from easy to achieve and poses a problem; most research in algebra teaching was mainly focused on the study of difficulties learning challenges encountered by students when moving from geometry to algebra. In this article we focus on the errors committed during the production of an activity in the geometric sense carried out by finding expressions which are represented as being a remarkable identity of dimension "3" the group experience "production on notebook" and the group “application on “GeoGebra” witness:

- They do not recognize the properties of the volumes of parallelized rectangles.
- They do not know how to use the “GeoGebra” applications which see themselves as a tool used to promote learning that manifests itself in three-dimensional space;
- They have difficulty operating on the patterns of figures; and they cannot chain together the calculated volumes to obtain the desired expressions.

Different interpretations of the sources of these difficulties are proposed. The long learning periods that students have completed in geometry come as an obstacle to their learning algebra. For example, Not correctly using different types of patterned figures such as hidden cubes, visible edges, etc., can result in incorrect interpretation of the drawing. Students have difficulty operating on patterned figures. This difficulty is the manifestation of a didactic break along the line of evolution from geometric to algebraic thinking.

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