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Research Article

Unravelling Aspects Of Mathematical Modeling Via Differential Equations

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ABSTRACT

Mathematical modeling serves as a pivotal tool in representing the behavior of real-world systems, allowing scientists and engineers to gain insights, make predictions, and solve complex problems. This paper explores the process of generating mathematical representations or models, validating them, understanding their utility, and recognizing their limitations. Before delving into these crucial aspects, it is imperative to understand the motivations behind mathematical modeling. Both engineers and scientists utilize mathematical modeling for practical reasons, aiming to address challenges and advance knowledge in their respective fields. Moreover, the joy of formulating and solving mathematical problems contributes to the appeal of mathematical modeling. Over the past few decades, the importance of mathematical modeling has been widely recognized, leading to a surge in research and publications in this domain. Mathematical modeling often involves differential equations when modeling situations with continuous variables and reasonable hypotheses about their rates of change. Depending on the number of dependent and independent variables, mathematical models may manifest as ordinary differential equations or systems of partial differential equations. This paper aims to provide insights into the fundamentals of mathematical modeling via differential equations, highlighting its significance across various disciplines.

Keywords: Mathematical modeling, Differential equations, Ordinary differential equations, Partial differential equations, Validation, Utility, Limitations, Interdisciplinary applications.

1. INTRODUCTION:

Mathematical modelling is an indispensable tool in the fields of science and technology, providing a methodical framework for comprehending the intricacies of mathematical representations of complex systems. Whether in engineering design, scientific inquiry, or problem-solving, mathematical modeling enables researchers to simulate, analyze, and predict the behavior of diverse phenomena. The essence of mathematical modeling lies not only in its practical utility but also in the intellectual challenge, satisfaction it offers to mathematicians, scientists, and engineers alike.
The evolution of mathematical modeling as a discipline has been profound, with its importance increasingly acknowledged over the past few decades. Today, mathematical modeling finds widespread application across every scientific and engineering domain, ranging from physics and biology to economics and environmental science. This surge in interest has led to a plethora of research works and publications, which either focus on specific disciplines, modeling techniques, or real-world situations.
At the heart of mathematical modeling lies the utilization of differential equations, particularly when dealing with continuous variables and their rates of change. Whether it's a single dependent variable evolving with respect to time or a complex system of multiple dependent and independent variables, differential equations provide the mathematical framework necessary for modeling dynamic processes. By formulating and solving these equations, researchers can gain valuable insights into the behavior of systems and phenomena under study.
In this paper, we aim to unravel the essence of mathematical modeling via differential equations, exploring process of model generation, validation, utilization, and recognizing its limitations. Through a systematic examination of fundamental principles and practical applications, we hope to elucidate the significance of

mathematical modeling in advancing scientific knowledge, solving real-world problems across diverse disciplines.

1.1 Linear Growth and Decay Models

(A) Exponential Growth Model

In the process of developing the nation's growth model, Malthus incorporated the subsequent three assumptions:

i. The population is sufficiently large.
ii. Population is homogeneous, that is, it is evenly spread over the living space.
iii. There are no limitations to growth i.e., no limitations of food, space 37 and so on. Population changes only by the occurrence of births and deaths.

Let x(t) denote the population size at time t, denoted as x (0) = x0, under the assumption that it is a positive integer.

On the assumption that population fluctuations are solely attributable to births and fatalities, immigration and emigration do not occur. Let B(t) and D(t) represent the quantities of births and fatalities at time t, respectively. The following equation represents the per capita fatality rate (m):

$$b = \frac{1}{x(t)}\frac{dD(t)}{dt}$$

$$m = \frac{1}{x(t)}\frac{dD(t)}{dt}$$

At any given moment, the growth rate for each member of the populace is denoted by

$$\frac{1}{x(t)}\frac{dB(t)}{dt} - \frac{1}{x(t)}\frac{dD(t)}{dt} = b - m$$

which gives,

$$\frac{1}{x(t)}\frac{d}{dt}(B - D) = b - m$$

therefore,

$$\frac{dx}{dt} = ax, where\ x(0) = x;$$

The distinction between the birth and mortality rates per capita (α ≥ b − m) is of significant importance and is referred to as the natural rate for expansion as well net growth rate.

Separating the variables, we get,

$$\frac{dx}{a} = dt$$

When both sides are integrated, the result is log x = αt + C, where C is a constant. At the outset, x = x_Q when t = 0, resulting in C = log x_0.

$$\therefore \log x = at + log x_0$$

Which means

$$x(t) = x_0 e^{-at}$$

If the net growth rate > 0, population x(t) grows exponentially without any bound as shown in fig. 1.1.

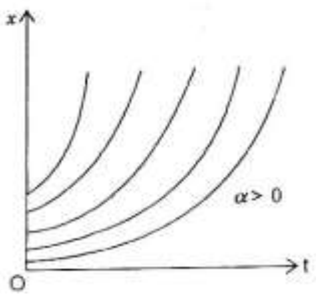

Fig.1.1

If α < 0, population x(t) decays exponentially and asymptotically decreases to extinction i.e., x(t) → 0 as t → ∞. Which is as shown in fig. 1.2.

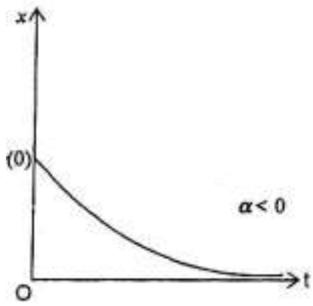

Fig.1.2

If α = 0, population x(t) = x0, remains constant which is as shown in fig. 1.3.

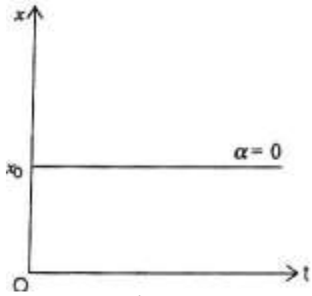

Fig.1.3

(B) Effects of immigration and Emigration on Population

Assuming constant rates of immigration and emigration, denoted by i and e, respectively, the rate of change in population can be computed as follows:

$$\frac{dx}{dt} = ax + i - e$$
$$= ax + \beta$$

On integrating, we get

$$\frac{1}{a} log(ax + \beta) = t + D$$

D=$\frac{1}{a} log(ax_0 + \beta)$

$$\therefore \frac{1}{a} log(ax + \beta) = t + \frac{1}{a} log(ax_0 + \beta)$$
$$\left[x_0 + \frac{\beta}{a}\right] e^{at} \frac{\beta}{a}$$
$$x(t) = x_0 e^{at} + \frac{\beta}{a}(e^{at} - 1)$$

1.2 Non-Linear Growth and Decay Models

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{K}\right)$$

Let x(0)=x_0

$$dt = \frac{1}{a} \frac{dx}{x\left(1 - \frac{x}{K}\right)}$$
$$dt = -\frac{K}{a} \frac{dx}{x(k - x)}$$
$$a\, dt = \left(\frac{1}{x} + \frac{1}{K - x}\right) dx$$

Integrating both sides

$$at = \int_{x_0}^{x} \left(\frac{1}{x} + \frac{1}{K - x}\right) dx$$
$$at = [logx - log(k - x)]_{x_0}^{x}$$
$$= [log - \frac{x}{K - x}]_{x_0}^{x}$$
$$log\left(\frac{x}{K - x}\right) - log\left(\frac{x_0}{K - x_0}\right)$$
$$at = log \frac{x(K - x_0)}{x_0(K - x)}$$

Taking antilog on both sides,

$$e^{at} = \frac{x(K - x_0)}{x_0(K - x)}$$
$$x_o(K - x)e^{at} = x(K - x_0)$$
$$x_0Ke^{at} - x_0xe^{at} = (K - x_0)$$
$$x_0Ke^{at} = x[K - x_0x + e^{at}]$$
$$x = \frac{x_0Ke^{at}}{K - x_o + x_0e^{at}}$$

The population size, denoted as the population density at time t, is represented by this equation. Undoubtedly, x(t) → K as t → ∞.
Differentiating equation w.r.t, 't

$$\frac{d^2x}{dt^2} = a\left[\frac{dx}{dt} - \frac{2x}{K}\frac{dx}{dt}\right]$$
$$= \frac{a}{k}[K - 2x]\frac{dx}{dt} = x\frac{a^2}{k^2}(k - 2x)(K - x)$$
$$\frac{d^2x}{dt^2} = \frac{a^2}{k^2}(k - 2x)(K - x)$$

If (K – 2x) > 0 i.e., K – x > x > 0

Then, $\frac{d^2x}{dt^2} > 0$
So that, the rate of increase of dt dx increases with time. This shows that there is an accelerated growth of the population in the range
0 < x < K/2
if K/2 < x < K, then K – 2x < 0 and K – x > 0, so that $\frac{d^2x}{dt^2} < 0$
The rate of $\frac{dx}{dt}$ decreases with time. Thus, there is a retarded growth of the population in the range K/2 < x < K

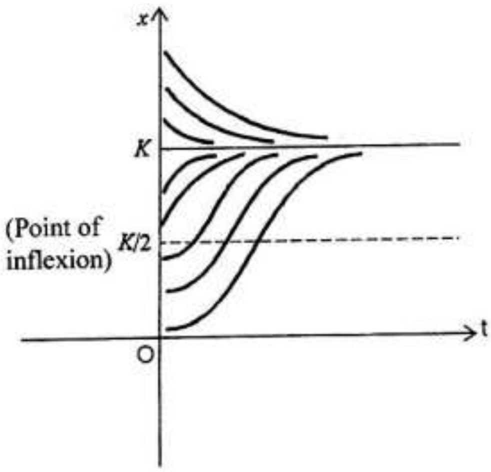

Fig.1.4

Now differentiating again w.r.t., x of both sides of, we get

$$\frac{d^3x}{dt^3} = \frac{a^2}{k^2}(k - 2x)(K - x) - x(K - 2x) - 2x(K - x)\frac{dx}{dt}$$
$$= \frac{a^2}{k^2}[(k - 2x)^2 - 2x(k - x)]\frac{dx}{dt}$$
$$= \frac{a^2}{k^2}[K^2 + 4x^2 - 4Kx - 2Kx + 2x^2]\frac{dx}{dt}$$
$$= \frac{xa^3}{k^3}[K^2 + 6x^2 - 6Kx](K - x)$$

The lines x(t) = 0 and x(t) = K give the equilibrium solution.
The point x = K/2 on the sigmoid is a point of inflexion in the population growth curve when half the final population size is reached

$$t_0 = \frac{1}{a}log\left(\frac{K}{x_0} - 1\right)$$

If x0 > K/2, there is no point of inflexion

1.3 Compartment Models

A vessel should be used to contain the ultimate volume of a liquid containing a constituent at a given time at an amount c(t). Permit a solution maintained in an overhead vessel at a constant concentration C to pass

through the chamber of solution at an equal pace R. After the solution and the mixture containing percentage c(t) have been thoroughly combined in the vessel, permit the mixture to depart at the same rate R while maintaining a constant volume V of the substance in the vessel.
Using the principle of continuity, we get

$$v(c(t+\Delta t)-c(t)=RC\Delta t-Rc(t)\Delta t+0(\Delta t)$$

$$v\frac{dc}{dt}+Rc=Rc$$

Integrating

$$c(t)=c(0)exp\left(-\frac{R}{V}t\right)+C\left(1-exp\left(1-\frac{R}{v}t\right)\right)$$

In due course, as time t approaches zero, c(0) transforms into C, causing the amount present in the vessel in question and the overhanging reservoir to be equivalent.

$$c(t)=c-(c-c_{0)}exp\left(-\frac{R}{v}t\right)$$

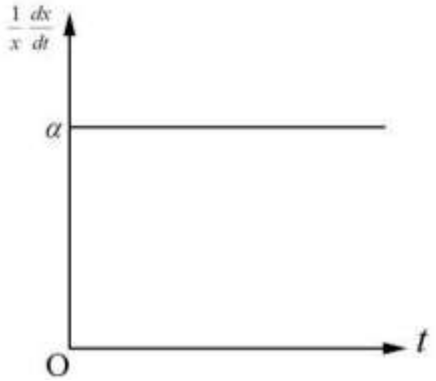

Fig 1.5

As per the continuity equations, if the rate R' is less than the rate R at which the answer exits the vessel, then

$$\frac{d}{dt}[(V_o+(R-\acute{R})t)c(t)]=RC-\acute{R}(ct)$$

2. Modelling at the First Order: Dynamic Through Ordinary Differential Equation
If a particle follows a straight line and travels a distance x in time i, then its velocity v can be calculated as dx/dt and its acceleration can be expressed as

$$\frac{dv}{dt}=\left(\frac{dv}{dx}\right)\left(\frac{dx}{dt}\right)=\frac{vdv}{dx}=\frac{dx}{dt^2}$$

Simple Harmonic Motion:
In this particular situation, the acceleration of a particle in a straight line is both constant and proportional to the particle's distance from the origin. Furthermore, the particle's trajectory remains in a consistent alignment with respect to its point of origin.

$$v\frac{dv}{dx}=-\mu x$$

Integrating

$$v^2=\mu(a^2-x^2),$$

where x = a represents the initial state of the particle. Equation (1)

$$\frac{dx}{dt}=-\sqrt{\mu}\sqrt{a^2-x^2}$$

The negative sign is employed because the magnitude of velocity increases as the value of x decreases.

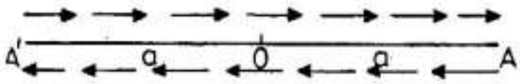

Fig.2.1
Integrating again and using the condition that at t = 0, x = a

$$x(t)=acos\sqrt{\mu}1$$

So that

$$v(t)=-a\sqrt{-\mu}\,sin\sqrt{\mu}\,\mathrm{t}$$

Both displacement and velocity in simple harmonic motion are periodic functions with periods 2π and μ.

The particle initiates its motion at point A with a velocity of zero, gradually approaches zero as its velocity increases, and finally reaches zero at time π 2/μ with a velocity of μa. The object continues along its path despite encountering a deceleration in velocity until it reaches point A'(0), where its velocity is once more zero. It then

commences a gradual ascent towards zero, attains zero with a velocity of μa, and returns to lying down at A shortly after a duration of 2π/μ. Subsequently, the periodic motion is repeated.
To illustrate the concept of SHM, let us examine an object of mass m that is affixed to one extremity of a perfectly flexible string, with the opposite extremity being connected to the fixed point 0 (see Figure 2.12). In vacuum, a particle moves under the influence of gravity.
Denote lo as the string's natural length and an as its extension at the point of equilibrium of the particle, such that according to Hooke's law:

$$mg = T_0 = \frac{\lambda}{l_0}$$

The equation of motion, which states that force acting on a particle in a particular direction equals mass × speed in that direction, yields

$$mv\frac{dv}{dx} = mg - T = mg - \lambda\frac{a+x}{l_0} = \frac{\lambda x}{l_0}$$

which, over time, produces a basic harmonic motion

$$\boldsymbol{2\pi\sqrt{\frac{a}{g}}}$$

The motion of gravity in a resisted medium Gravity causes a particle at a medium where resistance is proportional to velocity to descend. This is the momentum equation:

$$m\frac{dv}{dx} = mg - mkv$$

Integrating

$$V - v = Ve^{-kt}$$
$$v = V(1 - e^{-kt}),$$

with the intention of sustaining the velocity's ascent regarding the limiting velocity.
g/k as t → ∞. Replacing v by dx/dt,

$$\boldsymbol{x = Vt\frac{Ve^{-kt}}{k}} - \frac{V}{K}$$

2.Mathematical Modelling- Epidemic Dynamics

The quantities denoted as S(t) and I(t), respectively, are the number of susceptible and infected people. Initiate the system by introducing one susceptible and one infected individual, such that S(t) + I(t) = n+l, S(0) = n, and I(0) =
A system of linear equations whose solution is proportional to the sum of the numbers of susceptible and infected individuals describes a situation in which the number of susceptible individuals decreases at the same rate that the number of infected individuals increases.

$$\frac{ds}{dt} = -\beta SI, \frac{dl}{dt} = \beta SI,$$

So that

$$\frac{ds}{dt} + \frac{dl}{dt} = 0, S(t) + l(t) = constatnt = n + 1$$

and

$$\frac{ds}{dt} - \beta S(n + 1 - S),$$
$$\frac{dl}{dt}\beta t(n + 1 - 1)$$

Integrating,

$$\boldsymbol{S(t) = \frac{n(n+1)}{n+e^{(n+1)\beta t}}}, \boldsymbol{l(t) = \frac{(n+1)e^{(n+1)\beta}}{n+e^{(n+1)\beta t}}},$$

So that

$$\operatorname*{Lt}_{n\to\infty} S(t) = 0, \qquad \operatorname*{Lt}_{n\to\infty} l(t) = n + 1$$

(A)Susceptible-Infected-Susceptible (SIS) Model

$$\frac{dS}{dt} = -\beta SI + \Upsilon I, \qquad \frac{dI}{dt}\beta SI - \gamma I,$$

which gives

$$\frac{dI}{dt} = \beta(n + 1) - \gamma)I - \beta I^2$$

(B)SIS Model with Constant Number of Carriers

$$\frac{dI}{dt} = \beta(1 + C)S - \gamma I$$
$$= \beta C(n + 1) + \beta\left(n + 1 - C - \frac{v}{\beta}\right)I - \beta I^2$$

(C) Simple Epidence Model with Carriers

$$\frac{ds}{dt} = -\beta S(t) + \gamma I(t), \frac{dI}{dt} = \beta C(t)S(t) - \gamma I(t),$$
$$\frac{dC}{dt} = aC$$
$$S(t) + I(t) = S_0 + I_0 = N(say), c(t) = c_0 \exp(-at)$$
$$\frac{dI}{dt} = \beta_0 N exp(-at) - [\beta C_0 \exp(-at) + \gamma]I$$

(D) Model with Removal and Immigration

$$\frac{ds}{dt} = -\beta SI + \mu, \frac{dI}{dt} = \beta SI - \gamma I, \frac{dR}{dt} = \gamma \mathrm{I}.$$

2.1Compartment Models Using Systems of Ordinary Differential Equations

Denote the quantity of the drug present in the ith receptacle at time t as xi(t). It is assumed that the quantity transferable via the ith to the jth chamber (j ≠ i) within the time range (t, t + Δt) is denoted by kij, which stands for the transfer coefficient between the ith and jth compartments. The aggregate variation ∲xi at time Δt is determined by the quantity entering the ith chamber from other compartments, minus the quantity departing the ith chamber for other compartments, including the zeroth compartment, which represents the external system.

$$\Delta x_i = \sum_{j=o}^{n} k_{ij} x_i \Delta t + \sum_{j=1}^{n} k_{ij} x_i \Delta t + 0(\Delta t)$$

Dividing by Δt and proceeding to the limit as Δt → 0, we get

$$\frac{dx_i}{dt} = -x_i \sum_{j=1}^{n} k_{ij} + \sum_{j=1}^{n} k_{ij} x_i$$
$$= \sum_{j=1}^{n} k_{ji} x_j$$
$$k_{ii} = -\sum_{j=1}^{n} k_{ij}$$

In matrix notation,

$$\frac{dx}{dt} = kx,$$

Where

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad k = \begin{bmatrix} k_{11} & k_{21} & \dots & k_{n1} \\ k_{12} & k_{22} & \dots & k_{n2} \\ \dots & \dots & \dots & \dots \\ k_{1n} & k_{2n} & \dots & k_{nn} \end{bmatrix}$$

If X = B$e^{\gamma t}$, when B is a column matrix

$$\lambda B e^{\lambda t} = K B e^{\lambda t}$$

If B can be determined using this system of equations, it is consistent.

$$|K - \lambda I| = 0$$

I represent the n × n unit matrix. λ must therefore represent a unique value of the matrix K. Therefore, λ1, λ2, λ_n denote the eigenvalues.

$$Re(\lambda_1) \leq 0$$
$$Im(\lambda_i \neq 0 \; only \; if \; RI(\lambda_i) < 0$$

When the substance is administered at a consistent rate, the column vector D, consisting of components D1, D2, D_n, transforms into

$$\frac{dx}{dt} = KX + D$$

2.2 Modelling in Population Dynamics

(A) Prey-Predator Models

Assuming that the numbers of the prey and the predator organisms at time t are denoted by x(t) and y(t), the following first order regular differential equations are nonlinear systems:

$$\frac{dx}{dt} = ax = bxy = x(a - by, \;\; a, b > 0$$
$$\frac{dy}{dt} = -py + qxy = -y(p - qx), p, q > 0$$

Now $\frac{dx}{dt}, \frac{dy}{dt}$both vanish if

$$x = x_e \frac{p}{q}, y = y_e = \frac{a}{b}$$

If the initial ratios of prey and predator species are p/q and a/b, respectively, then these ratios will maintain over time.

.

$$\frac{dy}{dx} = \frac{-x(p - qx)}{x(a - by)}$$

$$\frac{a - by}{y} dy = -\frac{p - qx}{x} dx; x_0 = x(0), y_0 = y(0)$$

Integrating

$$\text{a In} \frac{y}{y_0} + p \text{ In } \frac{x}{x_0} = b(y - y_0) + q(x - x_0)$$

The distinct trajectory traverses every point within the initial x-y plane segment.

When we begin with (0, 0) or (p/q, a/b), point trajectories are produced. Using the initial values x = x0 and y = 0, we can determine that x improves while y remains at zero. Likewise, commencing with x = 0 and y = y_0, observe that x remains at zero and y diminishes. Affirmative axes of x and y therefore produce two line trajectories.

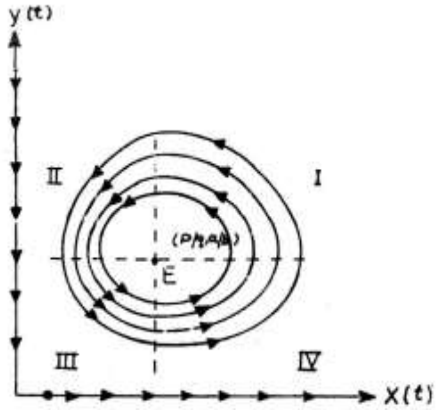

Fig.2.2

(B) Competition Models

The two species, represented by the population sizes x(t) and y(t), are in competition for the identical resources. Each species exhibits growth in the absence of the other, but its growth rate diminishes in its presence. The resulting system is composed of differential equations.

$$\frac{dx}{dt} = ax - bxy = bx\left(\frac{a}{b} - y\right)$$

$$\frac{dy}{dt} py - qxy = y(p - px) = qy\left(\frac{p}{q} - x\right)$$

$$\frac{dy}{dx} = \frac{y(p - qx)}{x(a - by)}$$

Integrating,

$$\text{a In} \frac{y}{y_0} - b(y - y_0) = p \text{ In } \frac{x}{x_0} - q(x - x_0)$$

The trajectory which passes through (p/q, a/b) is

$$\text{a in } \frac{by}{a} - by + a = p\, In\, \frac{qx}{p} - qx + p$$

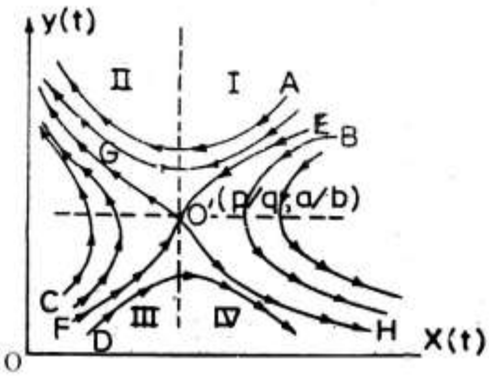

Fig.2.3

(C) Multi-species Models

The model under consideration is denoted by a system of differential equations.

$$\frac{dx_1}{dt} = a_1x_1 + b_{11}x_1^2 + b_{12}x_1x_2 + \cdots + b_{1n}x_1x_n$$
$$\frac{dx_2}{dt} = a_2x_2 + b_{21}x_2x_1 + b_{22}x_2^2 + \cdots + b_{2n}x_2x_n$$

The local stability of an equilibrium position x_10 x_20, x_n can be deduced by substituting

$$x_{1=}x_{10} + u_1, x_{2=}x_{20} + u_2, \ldots, x_n = x_{n0}u_n$$

and getting a system of linear differential equations

$$\frac{du_1}{dt} = c_{11}u_1 + c_{12}u_2 + \cdots + c_{1n}u_n$$
$$\frac{du_2}{dt} = c_{21}u_1 + c_{22}u_2 + \cdots + c_{2n}u_n$$

by neglecting squares, products and higher powers of u_i's. We can try the solutions $u_1 = A_1e^{\lambda t}, u_2 = A_2e^{\lambda t}, \ldots u_n = A_ne^{\lambda t}$ to get

$$\begin{vmatrix} c_{11}-\lambda & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{21} & c_{22-\lambda} & c_{23} & \vdots & c_{2n} \\ \cdots & \cdots & \cdots & & \cdots \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nn-\lambda} \end{vmatrix} = o$$

Therefore, stability can be achieved in the equilibrium position when all the real parts of the eigenvalues of the matrix [c_ij] are negative. The Routh-Hurwitz criterion specifies the prerequisites for this, namely that all the origins of

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \qquad a_0 > 0$$

negative real parts if and only if $T_0,\ T_1,\ T_2$, ... are positive where

$$T_0 = a_0, T_1 = a_1, T_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, T_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}$$
$$T_4 = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & 0 \\ a_5 & a_4 & a_3 & a_3 \\ a_7 & a_6 & a_5 & a_4 \end{vmatrix}$$

This assertion is only valid under the condition that a_i is larger than zero and either all even-numbered T_k or all odd-numbered T_k are positive. The (n–l) th degree equation, conversely, will satisfy this condition with all roots having negative real portions.

$$a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots - \frac{a_0}{a_1}a_3x^{n-4} - \cdots = 0$$

(D) Age-Structured Population Models

$$\frac{dx_1}{dt} = b_p1x_{p+1} + \cdots + b_{p+q}x_{P+q} - (d_1 + m_1)x_1$$
$$\frac{dx_n}{dt} = m_{n-1}x_{n-1} - d_nx_n;\ \ n = p + q + r$$
$$\frac{dX}{dt} = AX(t)$$

Each element along the diagonal of the provided matrix A is negative, whereas every element along the primary sub-diagonal is positive. Moreover, the initial row comprises q positive elements, whereas the remainder is composed entirely of zeros. The solution is denoted by the aforementioned equation.
X(t) = exp (At)X (Q)

2.3 Mathematical Modelling in Economics Using Systems of Ordinary Differential Equation of First Order

(A) Domar Macro Model

Consider S(t), I(t), and Y(t) to be the national income, savings, and investments at time t, respectively. Assuming savings and national income are proportional, S(t) = αY(t), where α is greater than zero.
Consequently, given that investment growth is proportional to the rate of national income expansion,

$$I(t) = \beta Y'(t), \beta > o$$

All savings are invested, so that

$$S(t) = I(t)$$

A system of three first-order ordinary differential equations is obtained in order to calculate S(t), Y(t), and I(t). Solving yields

$$Y(t) = Y(0)e^{\frac{at}{\beta}},\quad I(t) = aY(0)e^{\frac{at}{\beta}} = S(t)$$

(B) Domar First Debt Model

Presume the subsequent: Denoted as D(t) and Y(t), respectively, may symbolise the aggregate national debt and total national income.

The relationship between national income and the rate of change of national debt is such that D'(t) = αY (t). National income increases at a constant rate, so that Y'(t) = β

$$D(t) = D(0) + ay(0)t + \frac{1}{2}a\beta t^2$$
$$Y(t) = Y(0) + \beta t$$
$$\frac{D(t)}{Y(t)} = \frac{D(0) + ay(0)t + \frac{1}{2}a\beta t^2}{Y(0) + \beta t}$$

(C) Domar's Second Debt Model

Specifically, D(t)/Y(t) → as t approaches ∞. Thus, if the ratio of debt to income is not to increase indefinitely when debt increases at a rate proportional to income, then income must increase exponentially.
Y'(t) = βY(t)

$$Y(t) = Y(0)e^{\beta t}$$
$$D(t) = D(0) + \frac{\alpha}{\beta}y(0)(e^{\beta t} - 1)$$
$$\frac{D(t)}{Y(t)} = \frac{D(0)}{Y(0)e^{\beta t}} + \frac{\alpha}{\beta}(1 - e^{\beta t})$$

(D) Allen's Speculative Model

Denoting the demand, supply, and price of a commodity as d(t), s(t), and p(t), the following model can be derived:

$$\text{d(t)}=a_0 + a_1 p(t) + a_2 p'(t)$$
$$\text{s(t)}=\beta_0 + \beta_1 p(t) + \beta_2 p'(t)$$

The coefficients α0 and β2 in Allen's model represent the impact of conjecture. In the event of a price increase, supply decreases for the same reason that demand increases in anticipation of further price increases.
In regard to dynamic equilibrium
d(t) = s(t),

$$(\beta_2 - a_2)\frac{dp}{dt} + (\beta_1 - a_1)p(t) = a_0 - \beta_0$$
$$p(t) = p_e + (p(0) - p_e)e^{\lambda t}$$
$$p_e = \frac{a_0 - \beta_0}{\beta_1 - a_1}, \qquad \lambda = \frac{a_1 - \beta_1}{\beta_2 - a_2}$$

(E) Samuelson's Investment Model

Assume that K(t) signifies the capital and I(t) denotes the investment at time t.

$$\frac{dK}{dt} = I(t)$$

When capital is insufficient below a particular equilibrium level, the rate of investment increases proportionately to the shortfall, and when capital is excess above this equilibrium level, the rate of investment is declared, once more proportionate to the surplus, so that

$$\frac{dk}{dt} = I(t), \frac{dI}{dt} = -mk(t)$$
$$-mk(t) = \frac{dI}{ft} = \frac{dI}{dk}\frac{dk}{dt} = I\frac{dI}{dk}$$

Integrating

$$I^2 = m(k_0^2 - k^2)$$

So that

$$\frac{dk}{dt} = -\sqrt{m}\sqrt{k_0^2 - k^2}$$
$$\text{k(t)=k(0)cos}\sqrt{m}t$$
$$I(t) = -k(0)\sqrt{m}sin\sqrt{m}t$$

(F) Samuelson's Modified Investment Model

In this instance, excessive capital is not the only factor slowing down investment rate; high investment levels also contribute to this, as shown by above

$$\frac{dk}{dt} = I(t), \frac{dI}{dt} = -mk(t) - nI(t)$$
$$I\frac{dI}{dk} + mk(t) + nI(t) = 0$$

$$\frac{d^2k}{dt^2} + n\frac{dk}{dt} + mk = 0$$

(G) Stability of Market Equilibrium

Consider the price, supply, and demand of a commodity on the rth market denoted by p_r(t), s_r(t), and d_r(t), respectively. Then, Evan's price adjustment model mechanism suggests:

$$\frac{dP_r}{dt} = -\mu(s_r\text{-}d_r)$$

$$s_r\text{-}d_r = c_r + \sum_{s=1}^{n} d_{rs}P_s$$

$$\frac{dPLr}{dt} = -\mu_r(c_r + \sum_{s=1}^{n} d_{rs}P_s)$$

If P_{1e} , P_{2e} , . . . P_{ne} are the equilibrium prices in the n markets and

$$P_r = p_r - p_{re}$$

$$\frac{dP_r}{dt} = -\mu_r \sum_{s=1}^{n} d_{rs}P_s = \sum_{s=1}^{n} e_{rs}P_s$$

Therefore, stability of the equilibrium is ensured that every eigenvalue of the matrix E possesses negative real portions.

2.4 Models in Arms Race and Battles

(A) Richardson's Model for Arms Race

Let x(t) and y(t) denote the armaments expenditures of countries A and B, respectively. The term proportional to y characterises the tempo change dx/dt for country A's spending, since an increase in country B's expenditure on armaments results in a corresponding increase in country A's expenditure (90). Similar to that, it bears a term equal to (−x) since A's rate of arm expenditure is inhibited by its own arm expenditure. It might also include a clause that isn't related to the costs, based on shared mistrust or good intentions. Taking these things into account, Richardson provided the model

$$\frac{dx}{dt} = ay - mx + r, \frac{dy}{dt} = bx - ny + s$$

A position of equilibriumx_0, y_0, if it exists, will be given by

$$x_0 = \frac{as+nr}{mn-ab}, \quad y_0 = \frac{ms+br}{mn-ab}$$

(B) Lanchester's Combat Model

Given x(t) and y(t) as the advantages of both forces at war, and M and N as each person's fighting ability based on training, weaponry, and physical fitness, Lanchestel proposed that the decrease in one force's strength is proportionate to the increase in the other force's effective fighting strength.

$$\frac{dx}{dt} = -dyN, \qquad \frac{dy}{dt} = -axM$$

$$\frac{dx}{yN} = \frac{dy}{xM}$$

$$\frac{1}{x}\frac{dx}{dt} = \frac{1}{y}\frac{dy}{dt}$$

2.5 Mathematical Modelling in Dynamics

(A) Modelling in Dynamics

The primary aim of a particle motion analysis in a two-dimensional space is to ascertain the velocity components u(t) and v(t), alongside the particle's coordinates x(t) and y(t), with respect to a specific moment (t). In the context of three-dimensional motion, this method is also pertinent to the determination of the velocity components and coordinates x, y, z, u, v, and w. In order to ascertain the time (t) dependence of a solid object in three-dimensional natural space, the following twelve quantities are necessary:
It is delineated by six associated coordinates and velocities, in addition to six angular velocities and angles that encircle its centre of gravity.

The fundamental principle underlying equations of motion is that the force applied in a particular direction is equivalent to the product of the acceleration and mass acting in that direction. The implementation of systems comprised of second-order inequality equations is the consequence of this principle. It is possible, nevertheless, to reduce a second-order standard differential equation to two first-order regular differential equations by deriving acceleration and movement in relation to time and utilising the relationship between displacement and motion.

(B) Motion of a Projectile

At a velocity of V, a mass-m molecule accelerates from its inertial position in a vacuum to an angle of inversion with respect to the horizontal.
The equations that govern the motion of an object at time t are as follows: u(t) and v(t) represent the vertical and horizontal components of velocity, respectively.

$$m\frac{du}{dt} = 0 \quad m\frac{dv}{dt} = -mg$$

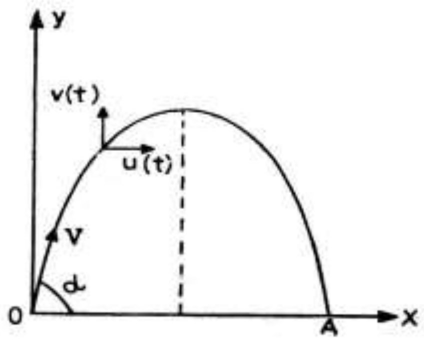

Fig.2.4

3.Mathematical Modelling Through-
Linear Differential Equations of Second Order
(A) Uniplanar Motion where the Accelerations Parallel to Fixed Axes Are Given

At a given instant, the positions of a particle denoted by the axes O_(x) and O_y be x and y, respectively, and its accelerations perpendicular to the axes be X and Y.
The equations of motion are then:

$$\frac{d^2x}{dt^2} = X$$
$$\frac{d^2y}{dt^2} = y$$

When each of those equations is integrated twice, two equations with four arbitrary constants are produced. The initial values of the initial conditions are used to ascertain these constants.

$$x, y. \frac{dx}{dt}, \frac{dy}{dt}$$

(B) Parabolic motion wider gravity, supposed constant, the resistance of the air being neglected.

Vertical acceleration is zero and the acceleration upward is −g when the y-axis is considered to be elongated vertically while the x-axis is considered to be horizontal. Consequently, the formulas for the equations of motion are modified to read as follows:

$$\frac{d^2x}{dt^2} = 0$$
$$\frac{d^2y}{dt^2} = -g$$

Integrating a second time:

$$x = At + B$$
$$y = -\frac{1}{2}gt^2 + Ct + D$$
$$x = ucos(a)t, and\ y = u\sin(a)\,t - \frac{1}{2}gt^2$$

Eliminating t, we have

$$\frac{y}{x} = \tan(a) - \frac{g}{2u^2cos^2(a)}$$

(C) Rectilinear Motion

The particle's equation of motion, taking into account Hooke's law and Newton's second law, is as follows:

$$m\frac{d^2x}{dt^2} = -K\frac{x}{L}$$

The elastic constant is denoted as k, and the original extent of the string is denoted as L.
.

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

which is a second-order linear differential equation. Its remedy is

$$x(t) = A_1e^{\lambda 1t} + Ae^{\lambda 2t}$$

When the force from outside is periodic with a period of 2π/w and the motion is simple fundamental with that period in its absence in the external force. The solution can be expressed as

$$x(t) = Acos(w_0t - a) + \frac{F}{2w_0}t\sin w_0$$

Periodically, the amplitude of the first term never surpasses |A| when w = wo. However, as time advances in a sequence with sin wot = ± 1, the magnitude of the second term approaches infinity.
Vibrating systems such as bridges, automobiles, aircraft, and ships are susceptible to injury when subjected to an external periodic force that coincides with their natural frequency. This is the rationale behind the

prohibition of soldiers marching in step while traversing a bridge. Nevertheless, the resonance phenomenon can also be utilised to one's benefit, as in the case of removing a vehicle from a ravine or uprooting trees. Beats manifest when the solution is the result of the merger of two sinusoidal oscillations with minimal period variation, as evidenced by the slight discrepancy between w and w_0.

(D) Electrical Circuits

The current i(t), measured in amperes, represents the rate at which it changes of charging q as it traverses the circuits in a continuous manner.

$$\frac{dq}{dt} = i(t)$$

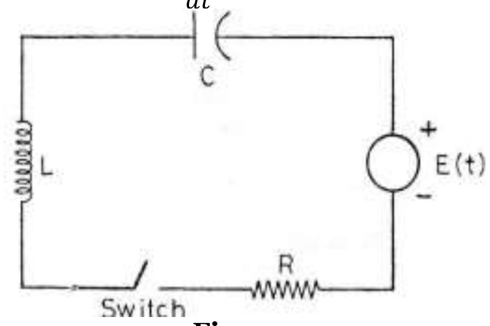

Fig.3.1

$$E_c = \frac{1}{C} q$$

The battery generates an E-volt voltage that exactly offsets each of these potential losses. As a result, the mathematical sum of the voltage drops surrounding a closed circuit is zero, in accordance with Kirchhoff's second law.

$$Ri + L\frac{di}{dt} + \frac{1}{c} q = E(t)$$

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{d}{c} i = \frac{dE}{dt}$$

$$L\frac{d^2 q}{dt^2} + R\frac{dq}{dt} + \frac{1}{c} q = E(t)$$

Linear differential equations are represented by both expressions, where i(t) and q(t) are constant coefficients.

3.1 Mathematical Modelling- Planetary Motions

The gravitational attractive force that regulates the motion of all planets is exerted by the Sun. By denoting the universal constant of gravitation G and the masses S and P, which correspond to the Sun and planet, respectively, the gravitational attraction forces exerted on both entities are determined to be GSP/r2, where r represents the distance separating the Sun and the planet. The acceleration of the Sun in relation to the planet is denoted by GP/r 2, as illustrated in Figure 4.3. Conversely, the acceleration of the planet in relation to the Sun is GS/r2. The acceleration of the planet with respect to the Sun is

$$\frac{G(s+p)}{r^2} = \mu r^2$$

Components of Velocity & Acceleration Vectors along Radial and Transverse Directions:

At the point P-to-Q transition of a particle, the displacement along the radius vector is

$$u = \underset{\Delta t \to 0}{\text{LT}} \frac{(r+\Delta r)cos\Delta\theta - r}{\Delta t}$$

$$= \underset{\Delta t \to 0}{\text{LT}} \frac{\Delta r}{\Delta t} = \frac{dr}{dt}$$

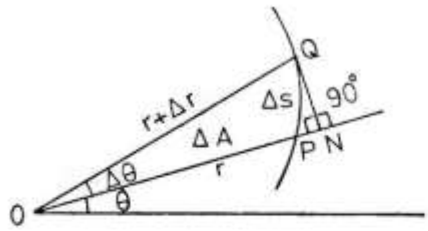

Fig.3.2

A displacement that is perpendicular to the radius vector.

$$=(r+\Delta r)\sin\Delta\theta$$

Velocity,

$$v = \underset{\Delta t \to 0}{\mathrm{LT}} \frac{(r+\Delta r) sin\Delta\theta}{\Delta t} = \underset{\Delta t \to 0}{\mathrm{LT}}\, r \frac{\sin \Delta\theta}{\Delta\theta} \frac{\Delta\theta}{\Delta t} r \frac{d\theta}{dt}$$

Acceleration,

$$\underset{\Delta t \to 0}{\mathrm{LT}} \frac{\Delta v \Delta\theta}{\Delta t} = \frac{du}{dt} - v\frac{d\theta}{dt}(r') - r\theta\ \theta$$

Therefore, the radial and transverse acceleration components are

$$r'' r\theta^2 and\ \frac{1}{2}\frac{d}{dt} r^2\theta'$$

Motion Under a Central Force:

$$m(r'' - r\theta^2) = -mF(r)$$
$$r'' - r\theta^2 = -F(r)$$

in order to derive a differential equation involving the variables θ and r. It is more practical to utilise u = 1/r rather than r.

$$r' = \frac{dr}{dt} = \frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{h}{r^2} = -h\frac{du}{d\theta}$$
$$\text{-}F(r) = -h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}h^2u^2\frac{d^2u}{d\theta^2} + u$$
$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}$$

In this context, F is readily representable as a function of u. The integration of this second-order differential equation will yield the relationship between θ and u or θ and r, i.e. the equation of the path a particle following a central force F per unit mass follows.

Motion Under the Inverse Square Law:

If the central force per unit mass is μ/r2 or μu 2

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}$$

When this linear equation is integrated, constant coefficients

$$\frac{h^2/u}{r} = \frac{L}{r} = 1 + ecos(\theta - a); h^2 = \mu L$$

Velocity V,

$$V^2 = r^2 + r^{2\theta'2} = \left(\frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt}\right) + \frac{1}{u^2}(hu)^2$$
$$= h^2\left(\frac{du}{d\theta}\right) + h^2u^2$$

by placing a focus at the centre of force, the force per unit mass can be expressed as

$$F = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$
$$\frac{h^2}{L}u^2 = \frac{\mu}{r^2}$$

Considering the elliptical trajectories of all planets around the Sun, it is logical to deduce that the principle regulating the attraction between the Sun and the planets must conform to the inverse square law.

Kepler's Laws of Planetory Motions:

It deduced fundamental principles regulating dynamics for two distinct planets with masses P1 and P2 and semi-axes of orbits a1, a2 from empirically exhaustive observations. This results in

$$\frac{T_1}{T_2} = \frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\frac{a_1^{3/2}}{a_2^{3/2}} = \frac{\sqrt{G(S+P_2)}}{\sqrt{G(S+1)}}\frac{a_1^{3/2}}{a_2^{3/2}}$$
$$\frac{T_1^2}{T_2^2} = \frac{s+P_2}{s+P_1}\frac{a_1^3}{a_2^3} = \frac{1+\frac{P_2}{s}}{1+\frac{P_1}{s}}\frac{a_1^3}{a_2^3}$$

According to the first law, each planet follows an elliptical trajectory around the Sun, wherein the Sun is situated at one of the foci of the ellipse. The second law of conservation of angular momentum states that the radius of the vector from the Sun to a planet traverses equal areas in equal time intervals. The cubes of the semimajor axes of planets' orbits are proportional to the squares of their orbital periods, according to the third law, also known as Kepler's harmonic law. The derivation of these laws from the gravitational inverse square law was a significant triumph of mathematical modeling, allowing for precise predictions and understanding of celestial phenomena.

3.2 Mathematical Modelling of Circular Motion and Motion of Satellites

(A) Circular Motion

Acceleration's transverse component equals

$$\frac{1}{r}\frac{d}{dt}(r^2\theta') = \frac{1}{a}\frac{d}{dt}(a^2\theta') = a\theta''$$

Thus the velocity is $a\theta'$ along the tangent and the acceleration has two components $a\theta''$ along the tangent and $a\theta'$ along the normal

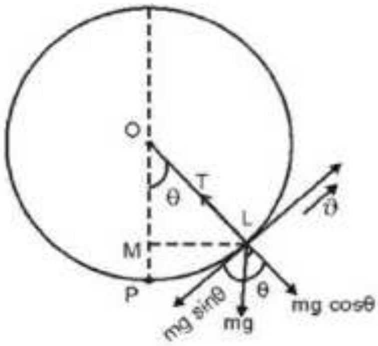

Fig.3.3

If a particle moves in a circle of radius a, its equations of motion are m $a\theta''$ = external force in the direction of the tangent m $a\theta'^2$ = external force in the direction of the inward normal.
Thus if a particle is attached to one end of a string, the other end of which is fixed and the particle moves in a vertical circle, the equations of motion are
m $a\theta'' = -msin\theta$
m $a\theta'^2 = T - mg\ cos\theta$
At the highest point θ = π and T = m $\frac{u^2}{a}$ −5mg. If $u^2 \geq$ 5 ag, the particle will move in the complete vertical circle again and again. However, if $u^2 <$ 5ag, tension will vanish before the particle reaches the highest point. When the tension vanishes, the particle begins to move freely under gravity and describes a parabolic path till the string again becomes tight and the circular motion is started again.

(B) Circular Motion of Satellites

If the Earth is of mass M and radius a and a satellite of mass m (« M) is projected from a point P at a height h above the Earth with velocity V at right angles to OP it will move under a central force Gm m/r^2 . Since the central force of a circular orbits is mv^2 /r, we get, if the path is to be circular,

$$\frac{mV^2}{a+h} = \frac{GmM}{(a+h)^2}$$

This provides the velocity of a satellite that is h metres above the Earth's surface and follows a circular orbit. The historical period is denoted by

$$T = \frac{2\pi(a+h)}{\sqrt{V}} = \frac{2\pi(a+h)}{\sqrt{ga}}(a+h)^{1/2} = \frac{2\pi}{\sqrt{ga}}(a+h)^{3/2}$$

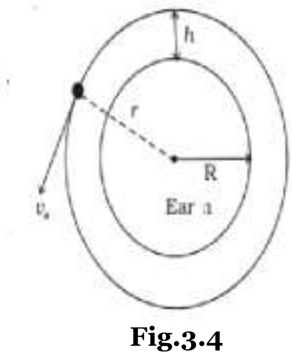

Fig.3.4

(C) Elliptic Motion of Satellites

If a satellite is projected at a height a + h above the center of the Earth with a velocity different from $\sqrt{ga}/\sqrt{a+h}$ or if it is not projected at right angles to the radius vector, the orbit will not be circular, but can be elliptic, parabolic or hyperbolic depending on V and the angle of projection.

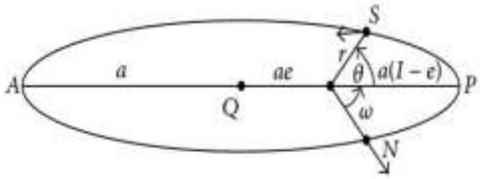

Fig.3.5

$$V^2 = \left(\frac{2}{a'(1-e)} - \frac{1}{a'}\right),\ a'(1+e) = a + h$$

$$V^2 = \mu\left(\frac{2}{a'(1-e)} - \frac{1}{a'}\right),\ a'(1+e) = a + h$$

$$V^2 = \frac{ga^2}{a+h}(1+e)$$

where V0 is the velocity required for a circular orbit for which e = 0. Thus if V > V0, the point of projection is nearest point of the orbit to the center of the Earth and if V < V0, this point is the furthest point
For the elliptic orbit, the time period is

$$e = \frac{h_{max} - h_{min}}{2a + h_{max} - h_{min}}$$

4. CONCLUSION:

In conclusion, mathematical modeling, particularly through the lens of differential equations, stands as a pivotal tool in scientific inquiry and engineering design. It offers a systematic approach to understanding the dynamics of complex systems, allowing researchers to simulate, analyze, and predict diverse phenomena. The evolution of mathematical modeling as a discipline has been profound, with its importance increasingly acknowledged over the past few decades. It finds widespread application across every scientific and engineering domain, contributing to advancements in physics, biology, economics, environmental science, and beyond.
At the heart of mathematical modeling lies the utilization of differential equations, providing the mathematical framework necessary for modeling dynamic processes involving continuous variables and their rates of change. Whether dealing with a single dependent variable evolving over time or a complex system of multiple dependent and independent variables, differential equations enable researchers to formulate and solve equations, gaining valuable insights into the behavior of systems and phenomena under study.
This paper has explored the process of generating mathematical models, understanding their utility, and recognizing their limitations. By examining fundamental principles and practical applications, we have unraveled the significance of mathematical modeling advancing scientific knowledge, solving real-world problems across diverse disciplines. Continued research and exploration in mathematical modeling promise to enhance our understanding of complex systems and drive innovation in science and engineering.

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