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Research Article

Harmonic Mean Cordial Labeling of Some Cycle Related Graphs

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ARTICLE INFO | **ABSTRACT**

All the graphs considered in this article are simple and undirected. Let $\boldsymbol{G} = (\boldsymbol{V(G)}, \boldsymbol{E(G)})$ be a simple undirected Graph. A function $\boldsymbol{f} : \boldsymbol{V(G)} \to \{\mathbf{1}, \mathbf{2}\}$ is called Harmonic Mean Cordial if the induced function $\boldsymbol{f^*}: \boldsymbol{E(G)} \to \{\mathbf{1}, \mathbf{2}\}$ defined by $\boldsymbol{f^*(uv)} = \left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor$ satisfies the condition $|\boldsymbol{v_f(i)} - \boldsymbol{v_f(j)}| \leq \mathbf{1}$ and $|\boldsymbol{e_f(i)} - \boldsymbol{e_f(j)}| \leq \mathbf{1}$ for any $\boldsymbol{i}, \boldsymbol{j} \in \{\mathbf{1}, \mathbf{2}\}$, where $\boldsymbol{v_f(x)}$ and $\boldsymbol{e_f(x)}$ denotes the number of vertices and number of edges with label $\boldsymbol{x}$ respectively and $\lfloor \boldsymbol{x} \rfloor$ is the floor function. A Graph $\boldsymbol{G}$ is called Harmonic Mean Cordial graph if it admits Harmonic Mean Cordial labeling. In this article, we have discussed Harmonic Mean Cordial labeling of Some Cycle Related Graphs.

Keywords: Harmonic Mean Cordial Labeling, Cycle, Helm graph, Web Graph, Total Graph

AMS Subject Classification (2020): 05C78, 05C76

1. Introduction

The notion of graph labeling in graph theory has garnered significant attention from scholars because of its wide-ranging and rigorous applications in domains such as communication network design and analysis, military surveillance, social sciences, optimization, and linear algebra. Various graph labelings are documented in the current body of literature. A dynamic survey of graph labeling by Gallian [1] is a condensed compilation of a lengthy bibliography of articles on the subject.

Let $G = (V(G), E(G))$ be a simple undirected Graph. Recall from [2] that a function $f : V(G) \to \{1,2\}$ is called Harmonic Mean Cordial if the induced function $f^*: E(G) \to \{1,2\}$ defined by $f^*(uv) = \left\lfloor \frac{2f(u)f(v)}{f(u)+f(v)} \right\rfloor$ satisfies the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{1,2\}$, where $v_f(x)$ and $e_f(x)$ denotes the number of vertices and number of edges with label x respectively and $\lfloor x \rfloor$ is the greatest integer less than or equals to x. A Graph G is called Harmonic Mean Cordial graph if it admits Harmonic Mean Cordial labeling. Motivated by the interesting results proved in [2, 3, 4] and on Root Cube Mean Cordial Labeling in [5], we have discussed HMC labeling of Harmonic Mean Cordial labeling of Some Cycle Related Graphs.

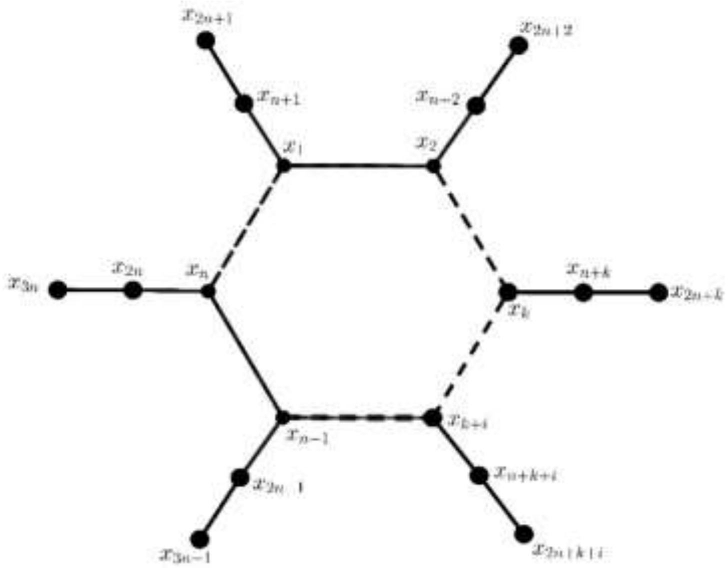
***Definition 1.1.* [1]** The armed crown is a graph in which path P_2 is attached at each vertex of cycle C_n by an edge. It is denoted by ACr_n where n is the number of vertices of cycle C_n.
***Definition 1.2.* [1]** The helm graph H_n is the graph obtained from a wheel W_n by attaching a pendent edge at each vertex of the cycle.
***Definition 1.3.* [1]** A web graph Wb_n is the graph obtained by joining the pendent vertices of a helm to form a cycle and then adding a single pendent edge to each vertex of this outer cycle.

Definition 1.4. **[1]** The Flower graph Fl_n is obtain from helm H_n by joining each pendent vertex to the apex of the helm.
Definition 1.5. **[1]** The Tadpole (Kite) is formed by joining the end point of a path P_n to a cycle C_m. It is denoted by $C_m@P_n$.
Definition 1.6. **[1]** Let G be a graph with two or more vertices then the total graph $T\ (G)$ of graph G is the graph whose vertex set is $V\ (G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G.

In Theorem 2.1 we have proved that Armed Crown graph ACr_n is HMC. The Helm graph H_n is HMC proved in Theorem 2.3. It is shown that The Web graph Wb_n is HMC in Theorem 2.5. In Theorem 2.7, it is proved that the Flower graph Fl_n is HMC. In theorem 2.9 and 2.11 we have derived the condition on natural numbers $m,\ n$ for which the Tadpole (Kite) graph $C_m@P_n$ is HMC. It is shown in Theorem 2.12 that the Total Path graph $T(P_n)$ is HMC.

2. Main Results

Theorem 2.1. The Armed crown ACr_n is HMC.
Proof. Let $G = (V, E) = ACr_n$ be the armed crown graph. Note that $|V| = |E| = 3n$. Let $V = \{x_1, x_2, \ldots, x_{3n}\}$ be the vertex set of G as shown in the following figure – 1:



ACr_n
***Figure* − 1**

Case 1: 3n is even

Define a labeling function $f : V(ACr_n) \to \{1,2\}$ as follows,

$$f(x_i) = \begin{cases} 2 \text{ if } 1 \le i \le \dfrac{3n}{2} \\ 1 \text{ if } \dfrac{3n}{2} + 1 \le i \le 3n \end{cases}$$

Then $v_f(1) = v_f(2) = \frac{n}{2}$ and $e_f(1) = e_f(2) = \frac{n+2}{2}$. So, we have $|v_f(1) - v_f(2)| = 0$ and $|e_f(1) - e_f(2)| = 0$.

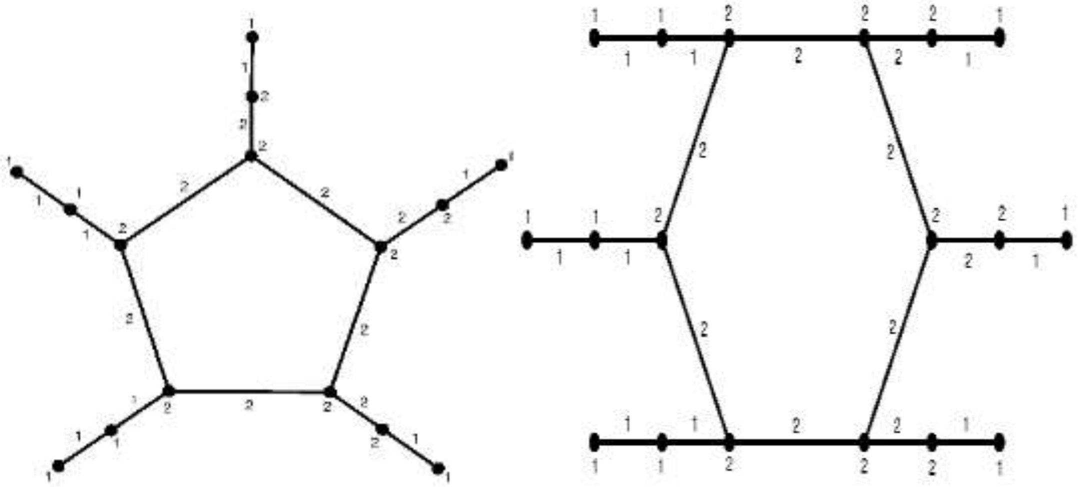
Case 2: 3n is odd

Define a labeling function $f : V(ACr_n) \to \{1,2\}$ as follows,

$$f(x_i) = \begin{cases} 2 \text{ if } 1 \le i \le \dfrac{3n+1}{2} \\ 1 \text{ if } \dfrac{3n+3}{2} \le i \le 3n \end{cases}$$

Then $v_f(1) = \frac{3n-1}{2}$, $v_f(2) = \frac{3n+1}{2}$ and $e_f(1) = \frac{3n-1}{2}, e_f(2) = \frac{3n+1}{2}$. So, we have $|v_f(1) - v_f(2)| = 1$ and $|e_f(1) - e_f(2)| = 1$.
Hence, ACr_n is HMC.

Illustration 2.2. HMC labeling of ACr_5 and ACr_6 is shown in following figure – 2.

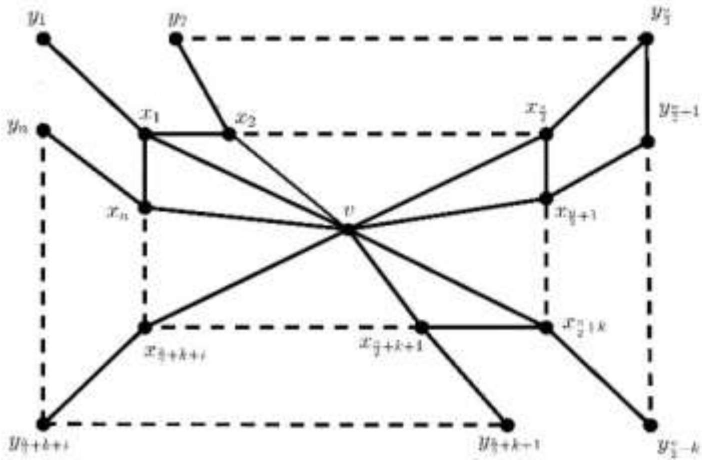


ACr_5 ACr_6

$\boldsymbol{Figure\ -2}$

Theorem 2.3. The Helm graph H_n is HMC.

Proof. Let $G = (V, E) = H_n$ be the Helm graph. Note that $|V| = 2n + 1$ and $|E| = 3n$. Let $V = \{v, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ be the vertex set of G as shown in the following figure – 2:



H_n

$\boldsymbol{Figure - 3}$

Case 1: n is even

Define a labeling function $f : V(H_n) \to \{1,2\}$ as follows,

$f(v) = 2,$

$f(x_i) = 2,$ if $1 \leq i \leq \frac{n}{2} + 1,$

$f(y_i) = 2,$ if $1 \leq i \leq \frac{n}{2} - 1,$

$f(x_i) = 1,$ if $\frac{n}{2} + 2 \leq i \leq n$ and

$f(y_i) = 1,$ if $\frac{n}{2} \leq i \leq n.$

Then $v_f(1) = n$, $v_f(2) = n + 1$ and $e_f(1) = e_f(2) = \frac{3n}{2}$. So, we have $|v_f(1) - v_f(2)| = 1$ and $|e_f(1) - e_f(2)| = 0$.

Case 2: n is odd

Define a labeling function $f : V(H_n) \to \{1,2\}$ as follows,

$f(x) = 2,$

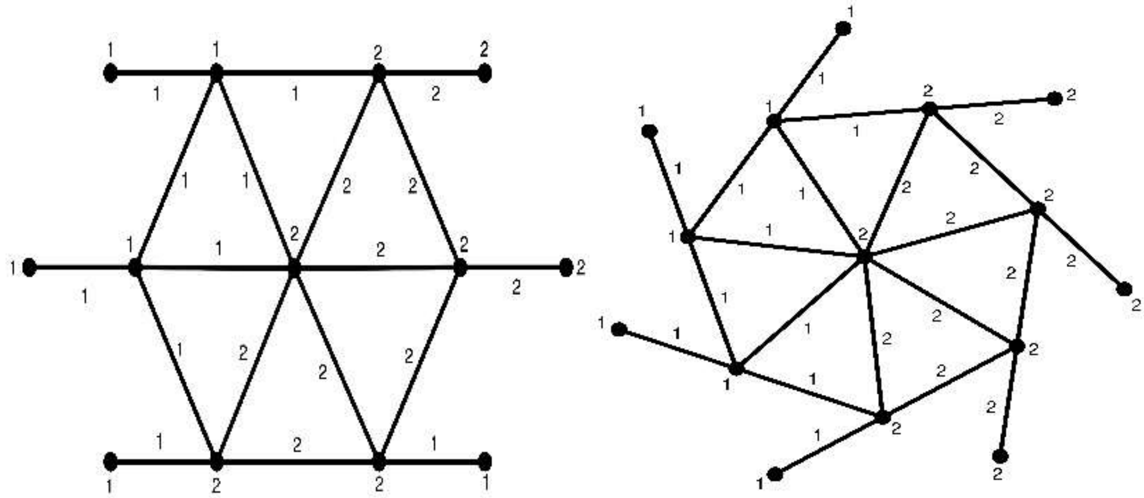
$f(x_i) = 2,$ if $1 \leq i \leq \frac{n+1}{2},$

$f(y_i) = 2,$ if $1 \leq i \leq \frac{n-1}{2},$

$f(x_i) = 1,$ if $\frac{n+3}{2} \leq i \leq n$ and

$f(y_i) = 1,\;$ if $\frac{n+1}{2} \leq i \leq n$.
Then $v_f(1) = n\,, v_f(2) = n+1$ and $e_f(1) = \frac{3n+1}{2}$, $e_f(2) = \frac{3n-1}{2}$. So, we have $|v_f(1) - v_f(2)| = 1$ and $|e_f(1) - e_f(2)| = 1$.
Hence, The Helm graph H_n is HMC.

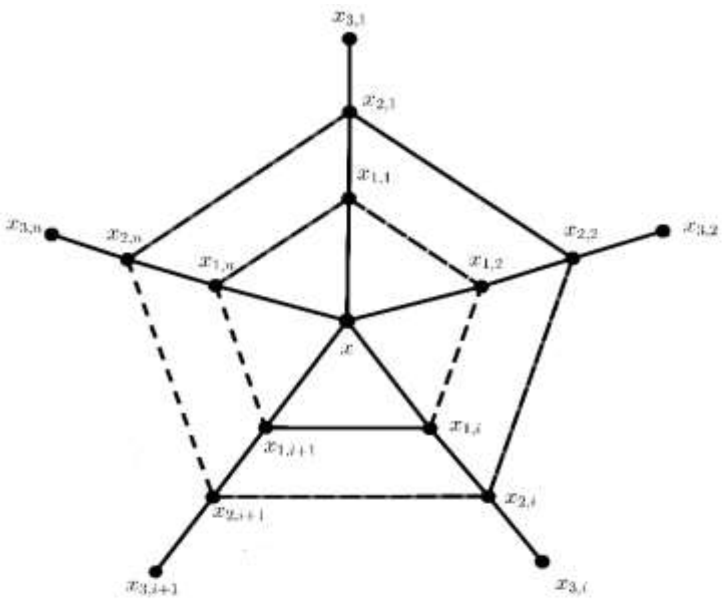
Illustration 2.4. HMC labeling of H_6 and H_7 is shown in following figure – 4.



H_6 H_7

***Figure* – 4**

Theorem 2.5. The Web graph Wb_n is HMC.
Proof. Let $G = (V, E) = Wb_n$ be the Web graph. Note that $|V| = 3n+1$ and $|E| = 5n$. Let $V = \{x, x_{1,1}, x_{1,2}, \ldots, x_{1,n}, x_{2,1}, x_{2,2}, \ldots, x_{2,n}, x_{3,1}, x_{3,2}, \ldots, x_{3,n}\}$ be the vertex set of G as shown in the following figure – 5:



Wb_n
***Figure* – 5**

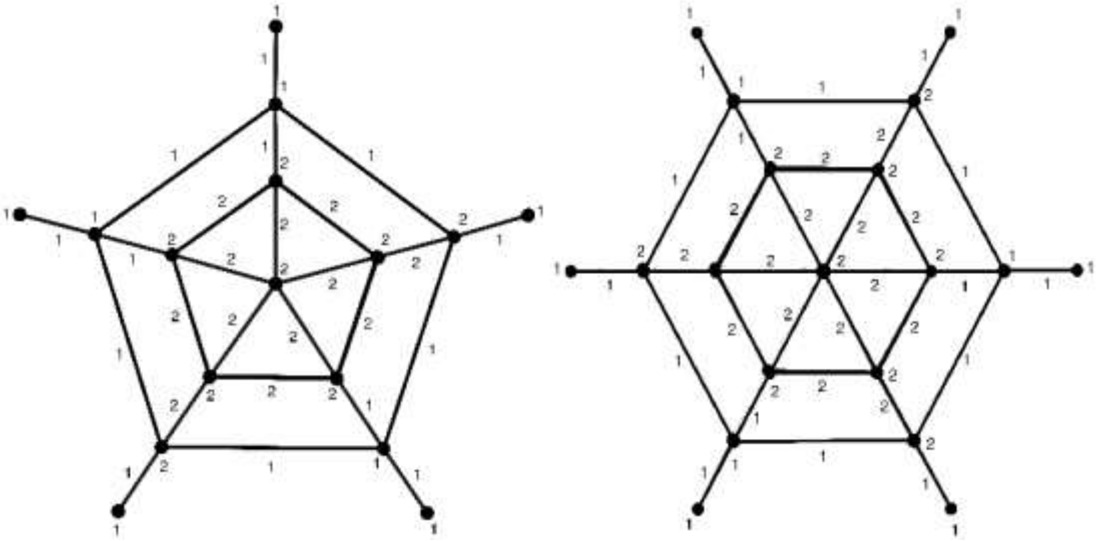
Case 1: $3n + 1$ is even
Define a labeling function $f: V(Wb_n) \to \{1,2\}$ as follows,
$f(x) = 2$,
$f(x_{1,i}) = 2,\;$ if $1 \leq i \leq n$,
$f(x_{2,i}) = 2,\;$ if $1 \leq i \equiv 0 \pmod 2 \leq n$,

$f(x_{2,i}) = 1,$ if $1 \leq i \equiv 1 \ (\text{mod } 2) \leq n, f(x_{3,i}) = 1,$ if $1 \leq i \leq n$. Then $v_f(1) = v_f(2) = \frac{3n+1}{2}$ and $e_f(1) = \frac{5n+1}{2}$, $e_f(2) = \frac{5n-1}{2}$. So, we have $|v_f(1) - v_f(2)| = 0$ and $|e_f(1) - e_f(2)| = 1$.

Case 2: $3n + 1$ is odd

Define a labeling function $f: V(Wb_n) \rightarrow \{1,2\}$ as follows,
$f(x) = 2,$
$f(x_{1,i}) = 2,$ if $1 \leq i \leq n,$
$f(x_{2,i}) = 2,$ if $1 \leq i \equiv 1 \ (\text{mod } 2) \leq n,$
$f(x_{2,i}) = 1,$ if $1 \leq i \equiv 0 \ (\text{mod } 2) \leq n,$
$f(x_{3,i}) = 1,$ if $1 \leq i \leq n$. Then $v_f(1) = \frac{3n}{2}$, $v_f(2) = \frac{3n+2}{2}$ and $e_f(1) = e_f(2) = \frac{5n}{2}$. So, we have $|v_f(1) - v_f(2)| = 1$ and $|e_f(1) - e_f(2)| = 0$.
Hence, The Web graph Wb_n is HMC.

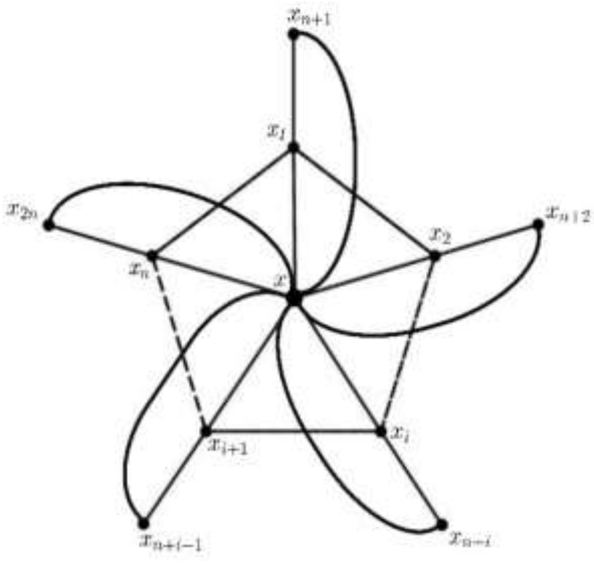
Illustration 2.6. HMC labeling of H_6 and H_7 is shown in following figure – 6.



Wb_5 Wb_6

Figure – 6

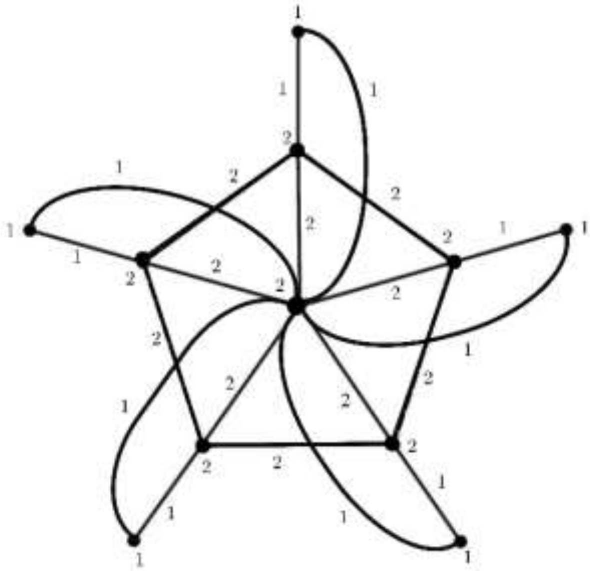
Theorem 2.7. The Flower graph Fl_n is HMC.
Proof. Suppose $G = (V, E) = Fl_n$ be the Flower graph. Note that $|V| = 2n + 1$ and $|E| = 4n$. Let $V = \{x, x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{2n}\}$ be the vertex set of G as shown in the following figure – 7:



Fl_n
Figure – 7

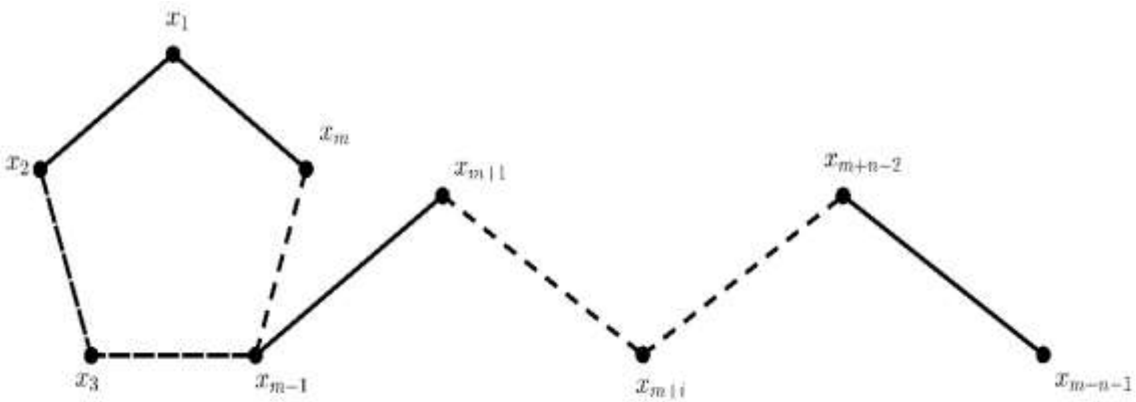
Define a labeling function $f : V(Fl_n) \to \{1,2\}$ as follows,
$f(x) = 2$,
$f(x_i) = 2$ if $1 \le i \le n$ and
$f(x_i) = 1$ if $n+1 \le i \le 2n$.
Then $v_f(1) = n$, $v_f(2) = n+1$ and $e_f(1) = e_f(2) = 2n$. So, we have $|v_f(1) - v_f(2)| = 1$ and $|e_f(1) - e_f(2)| = 0$.
Hence, Flower graph Fl_n is HMC

Illustration 2.8. HMC labeling of Fl_5 is shown in following figure – 8.



Fl_5
***Figure* – 8**

Theorem 2.9. The Tadpole $C_m@P_n$ is HMC for $m+n$ is even or $m+n$ is odd and $m < n$.
Proof. Let $G = (V, E) = C_m@P_n$ be the tadpole graph. Note that $|V| = |E| = m+n-1$. Let $V = \{x_1, x_2, \dots, x_{m+n-1}\}$ be the vertex set of G as shown in the following figure – 9:



$C_m@P_n$
***Figure* – 9**

Case 1: $m+n$ is even
Define a labeling function $f: V(C_m@P_n) \to \{1,2\}$ as follows,

$$f(x_i) = \begin{cases} 2 & \text{if } 1 \le i \le \frac{m+n}{2} \\ 1 & \text{if } \frac{m+n+2}{2} \le i \le m+n-1 \end{cases}$$

Then $v_f(1) = \frac{m+n-2}{2}$, $v_f(2) = \frac{m+n}{2}$ and $e_f(1) = \frac{m+n-2}{2}$, $e_f(2) = \frac{m+n}{2}$. So, we have $|v_f(1) - v_f(2)| = 1$ and $|e_f(1) - e_f(2)| = 1$.

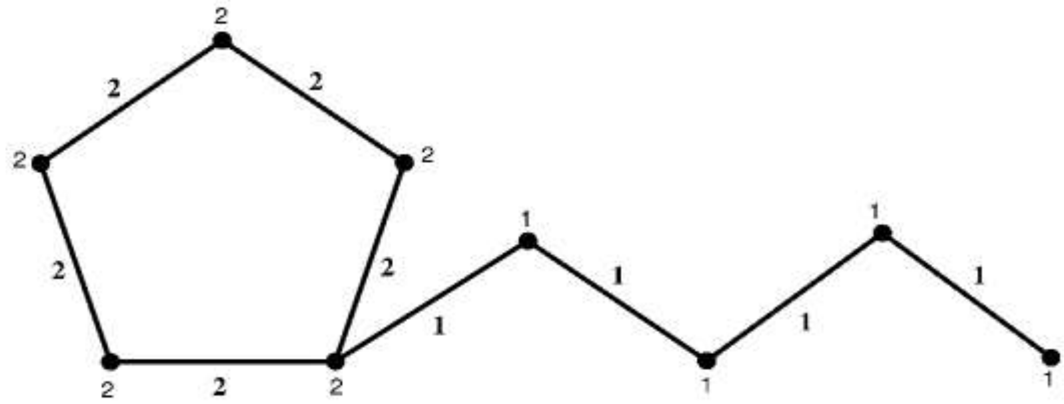
Case 2: $m+n$ is odd and $m < n$ $\forall m, n \in \mathbb{N}$
Define a labeling function $f: V(C_m@P_n) \to \{1,2\}$ as follows,

$$f(x_i) = \begin{cases} 2 \text{ if } 1 \leq i \leq \frac{m+n-1}{2} \\ 1 \text{ if } \frac{m+n+1}{2} \leq i \leq m+n-1 \end{cases}$$

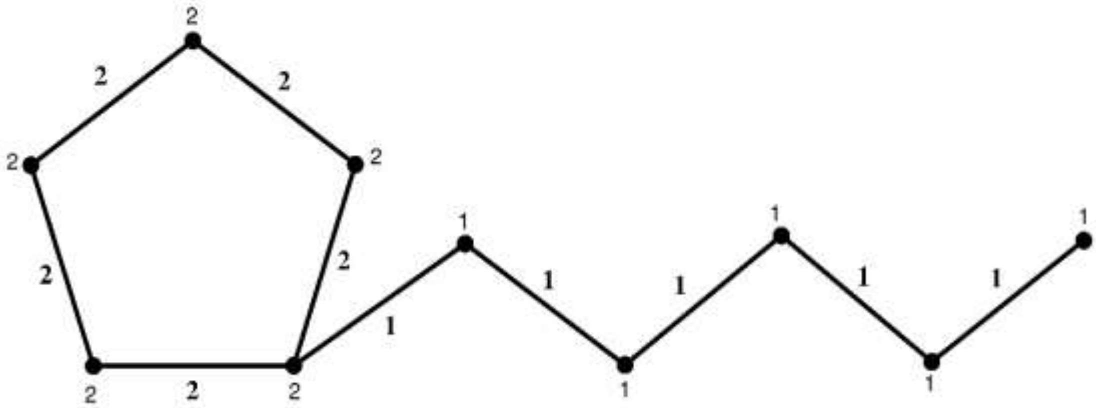
Then $v_f(1) = v_f(2) = \frac{m+n-1}{2}$ and $e_f(1) = e_f(2) = \frac{m+n-1}{2}$. So, we have $|v_f(1) - v_f(2)| = 0$ and $|e_f(1) - e_f(2)| = 0$.

Hence, The Tadpole $C_m@P_n$ is HMC for $m+n$ is even or $m+n$ is odd and $m < n$.

Illustration 2.10. HMC labeling of $C_5@P_5$ and $C_5@P_6$ is shown in following figure – 10.



$C_5@P_5$

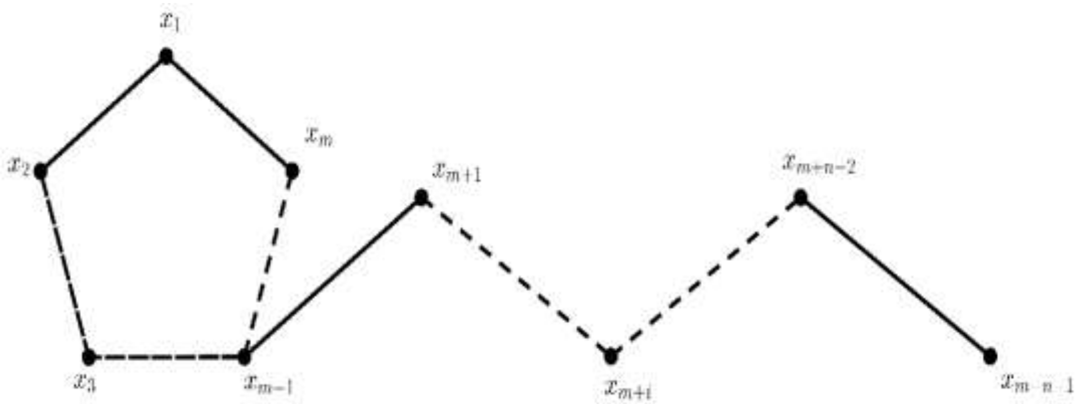


$C_5@P_6$

$\boldsymbol{Figure\ -10}$

Theorem 2.11. The Tadpole $C_m@P_n$ is not HMC for $m+n$ is odd and $m > n$.

Proof. Let $G = (V, E) = C_m@P_n$ be the tadpole graph. Note that $|V| = |E| = m+n-1$(even). Let $V = \{x_1, x_2, \ldots, x_{m+n-1}\}$ be the vertex set of G as shown in the following figure – 11:



$C_m@P_n$

$\boldsymbol{Figure - 11}$

Suppose that G is HMC. Then we have $v_f(1) = v_f(2) = \frac{m+n-1}{2}$. Now, we consider the following cases.

Case 1: Vertex x_{m-1} has labeling 1

Note that we have at least two edges whose one end vertex is 1 and another end vertex is 2 in C_m. Suppose that there exist r −edges whose one end vertex is 1 and another end vertex is 2. Then we have $e_f(1) \geq \frac{m+n+r-1}{2}$, $e_f(2) \leq \frac{m+n-r-1}{2}$. Note that $|e_f(1) - e_f(2)| \geq r > 1$.

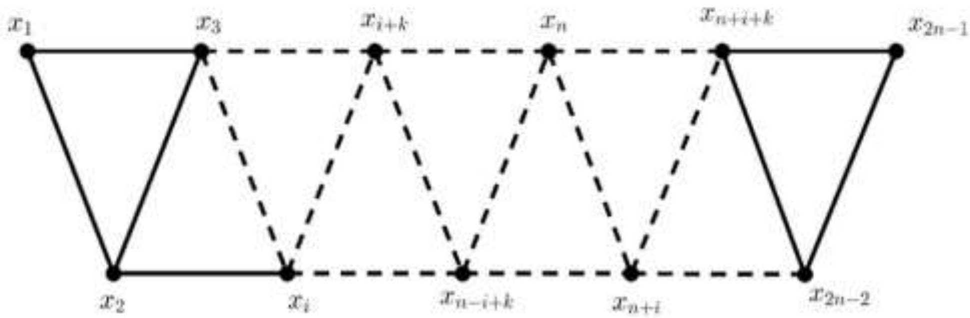
Case 2: Vertex x_{m-1} has labeling 2

Subcase 2.1: All the vertices of path have label 2. Then in C_m we have $\frac{m+n-1}{2} - (n-1) = \frac{m-n+1}{2}$ vertices of label 2. Hence $e_f(2) \leq n + \frac{m-n+1}{2} - 1 = \frac{m+n-1}{2}$ and $e_f(1) \geq \frac{m+n-1}{2} + 1 = \frac{m+n+1}{2}$. So, $|e_f(1) - e_f(2)| \geq n > 1$. we have $|e_f(1) - e_f(2)| > 1$.

Subcase 2.2: All the vertices of path have label 1. Then in C_m we have $\frac{m+n-1}{2}$ vertices of label 2. So, we have $m - \left(\frac{m+n-1}{2}\right) = \left(\frac{m-n+1}{2}\right) \geq 1$ vertices with labeled 1 in C_m. So, in this case we have at least two edges in C_m with one end vertex has labeled 1 and another end vertex has labeled 2. Also, edge $x_{m-1}x_{m+1}$ has one end vertex has labeled 1 and another end vertex have labeled 2. So, we have at least three edges whose one end vertex has labeled 1 and another end vertex have labeled 2. Suppose that there exist r −edges whose one end vertex is 1 and another end vertex is 2. Then we have $e_f(1) \geq \frac{m+n+r-2}{2}$, $e_f(2) \leq \frac{m+n-r}{2}$. Note that $|e_f(1) - e_f(2)| = r - 1 > 1$.

Subcase 2.3: If some of the vertices of labeled 2 are in P_n and some of the vertices of labeled 1 in C_m Then in C_m we have at least two edges with one end vertex has labeled 2 and another end vertex has labeled 1 and in P_n we have at least one edge with one end vertex has labeled 2 and another end vertex has labeled 1. So, we have at least three edges whose one end vertex has labeled 1 and another end vertex have labeled 2. Suppose that there exist r −edges whose one end vertex is 1 and another end vertex is 2. Then we have $e_f(1) \geq \frac{m+n+r-2}{2}$, $e_f(2) \leq \frac{m+n-r}{2}$. Note that $|e_f(1) - e_f(2)| = r - 1 > 1$.
Hence, The Tadpole $C_m@P_n$ is not HMC for $m + n$ is odd and $m > n$.

Theorem 2.12. The total path graph $T(P_n)$ is HMC.
Proof. Let $G = (V, E) = T(P_n)$ be the total path graph. Note that $|V| = 2n - 1$ and $|E| = 4n - 5$. Let $V = \{x_1, x_2, \ldots, x_{2n-1}\}$ be the vertex set of G as shown in the following figure – 12:



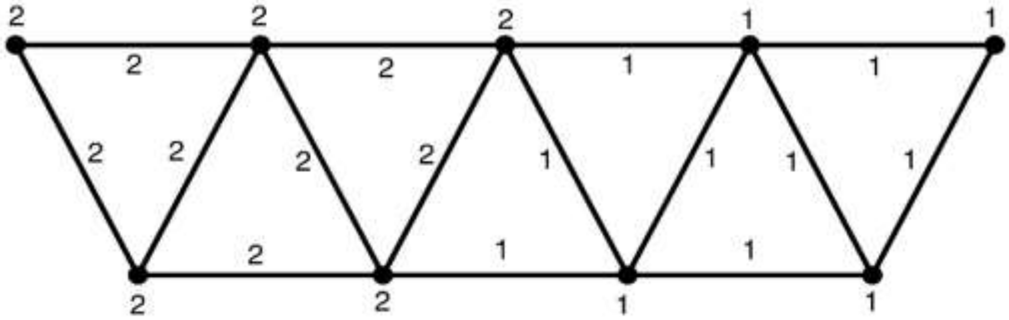
$T(P_n)$
***Figure* − 12**

Define a labeling function $f : V(T(P_n)) \rightarrow \{1,2\}$ as follows,

$$f(x_i) = \begin{cases} 2 & \text{if } 1 \leq i \leq n \\ 1 & \text{if } n+1 \leq i \leq 2n-1 \end{cases}$$

Then $v_f(1) = n - 1,\ v_f(2) = n$ and $e_f(1) = 2n - 2, e_f(2) = 2n - 3$. So, we have $|v_f(1) - v_f(2)| = 1$ and $|e_f(1) - e_f(2)| = 1$.
Hence, the total path graph $T(P_n)$ is HMC.

Illustration 2.13. HMC labeling of $T(P_5)$ is shown in following figure – 13.



$T(P_5)$
$\boldsymbol{Figure - 13}$

4. Conclusion

In this article we have proved that Armed Crown graph ACr_n, Helm graph H_n, Web graph Wb_n, Flower graph Fl_n , Total Path graph $T(P_n)$ are HMC. Also, we have derived the condition on natural numbers m and n for which Tadpole (Kite) $C_m@P_n$ is HMC.

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