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Research Article

On Some Simple Eulerian Lattices

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ABSTRACT

In this paper, we prove that $CS(L)$under set inclusion relation is Eulerian whenever L is a simple Eulerian lattice for some known simple Eulerian latticeslike Q, the dual of the face lattice of a cube, R, the face lattice of an icosahedron ,S(Q) and so on. We also prove that CS(L) is Eulerian for a dual simplicial Eulerian lattice.

Keywords: Convex sublattice, Eulerian lattice, Simplicial lattice, Simple, Face lattice. Mathematics Subject Classification 2020: 03G10,06C05,06C10

1. Introduction

K.M. Koh [2] began the study on the lattice of convex sublattices of a given lattice with respect to the set inclusion relation.

A sublattice C of a lattice L is said to be convex if a, $b \in C$, $c \in L$, a ≤ c ≤ b imply that $c \in C$. Let CS(L) be the family of all convex sublattices of L, including the empty set. Then CS(L), partially ordered by inclusion, forms an atomistic algebraic lattice. He had proved that CS(L) is distributive if and only if $L \cong L_n$,n = 1,2 where L_n denotes the chain of n elements. When L is a finite lattice and he has given a characterization for CS(L) to be lower semi modular as CS(L) is lower semi modular if and only if L is a chain. He had also proved that when L satisfies ACC and DCC, then L is relatively complemented if and only if CS(L) is relatively complemented. As any Eulerian lattice L is relatively complemented[8],by this result we infer that CS(L) is also relatively complemented. Now a natural question is whether CS(L) is Eulerian whenever L is Eulerian. Many authors have attempted to solve this problem. For example,. Dr.A.Vethamanickam and Dr.R.Subbarayan[14], have proved that $CS(B_n)$ is Eulerian when B_n is a Boolean algebra of rank n. In 2011, Sheeba Merlin.G and Vethamanickam.A[12] have proved the same for Eulerian lattices $S(B_n)$, $S(C_n)$ and $S_m(B_n)$.But they have done it only for some particular Eulerian lattices.

In the thesis of K.E.Usha[7], one open question was raised as to whether CS(L) is simple if L is an Eulerian lattice. By remark 3.4.2 in Usha's thesis we infer that to decide whether CS(L) is simple whenever L is Eulerian reduces to the problem of proving whether CS(L) is Eulerian whenever L is a simple Eulerian lattice. Therefore, the problem will be completely solved if we can prove that CS(L) is Eulerian whenever L is a simple Eulerian lattice. But one bottleneck in this attempt is we do not yet have a complete list of simple Eulerian lattices. The only known simple Eulerian lattices so far are the two element chain B_1, the face lattice C_n of the polygon of n sides n > 3, a lattice of the form $S(L) = \left(B_2 \times L\right) \cup \{\hat{0}, \hat{1}\}$, where L is an Eulerian lattice [13], $S_g(L_1,L_2,\ldots,L_n)$[13], where L_i's ,i = 1,2,3,…,n are Eulerian, $D_r(L) = \left(\dot{\bigcup}_{i=1}^{r} \overline{L_i}\right) \cup \{0,1\}$, $where\ \overline{L_i} = L_i \setminus \{0,1\}$ where each L_i is an Eulerian lattice of same rank and $\dot{\bigcup}$ stands for disjoint union [13], and some strongly uniform non-Boolean Eulerian lattices of rank ≤ 5 found in [8] . In this paper we prove that CS(L) is Eulerian for each of the above lattices L.

The lattice of all convex sublattices of a two element chain is B_2, the Boolean algebra of rank 2 and it is Eulerian. CS[S(L)] is Eulerian have been proved for some particular Eulerian lattices L,viz.,B_n, C_m by Sheeba Merlin .G and Vethamanickam. A [12] , S(B_n)[9] and $B_n \times C_m$ [10] by Usha Nirmala Kumari.K.E. and Vethamanickam.A. For a general Eulerian lattice L, proving CS[S(L)] to be Eulerian is an open problem. In the following sections we provide proofs for the remaining lattices mentioned in the previous paragraph.

2. Preliminaries

To prove this result, we need the following definitions and theorems.

Definition(simple)

A lattice L is said to be simple if it has no non- trivial congruences. For example,

1. The lattice M_3 is simple which is not Eulerian. Rank 2 Eulerian lattice is not simple.
2. The lattice $C_n, n \geq 4$ is simple.

Definition(Relatively complemented)

A lattice with 0 and 1 is said to be relatively complemented, if every interval of L is complemented. For example, $C_n, n \geq 3$ is relatively complemented.

Definition (Simplicial)

Let P be a poset with 0. P is said to be simplicial if for every element $t \in P$, $[0,t]$ is Boolean.
Dual simplicial poset is defined dually. Definition(r-simplicial)

A lattice L of rank $d \geq r$ is said to be r-simplicial, if $[0,x]$ is Boolean, for all elements x of rank r.

Definition(Strongly uniform)

A lattice L is said to be strongly uniform, if for every two elements x and y in L of the same rank, the upper intervals $[x,1]$ and $[y,1]$ are isomorphic.

Definition(Mobius function)

Let P be a finite poset, The Mobius function μ is an integer- valued function defined on $P \times P$ by the formulae:

μ(x, x) = 1, for x $\in$ P

$$\mu(x, y) = \begin{cases} 0, & if\ x \not\leq y \\ -\sum_{x \leq z < y} \mu(x,z), & if\ x < y \end{cases}$$

Definition(Graded)

A lattice L is said to be graded if all its maximal chains have same length.

Definition(Height of an element)

The height of an element a of a lattice L, denoted by ht(a) is the length of the longest maximal chain in (0,a]
Definition(Eulerian lattice)

A finite graded poset P is said to be Eulerian, if its Mobius function assumes the value $\mu(x, y) = (-1)^{l(x,y)}$ for all x ≤ y in P, where $l(x, y) = ht(y) - ht(x)$.

Example. Every Boolean algebra of Rank n is Eulerian and the lattice C_4 is a non modular Eulerian lattice.

Definition[5]

$$S(L) = \left(\overline{B_2} \times \overline{L}\right) - \{(1,1)\}, \overline{B_2} \cong B_2 \times \{1\}\ and\ \overline{L} \cong L \times \{1\}$$

where B_2 is the Boolean lattice of rank 2.

Remark

Face lattice of a polytope need not be simple.

For example, B_3 is not simple, B_3 is the face lattice of a triangle.

3. Simplicity of some known non-Boolean strongly uniform Eulerian lattices Theorem 3.1

The dual of the face lattice Q of a cube(hexahedron) is simple.

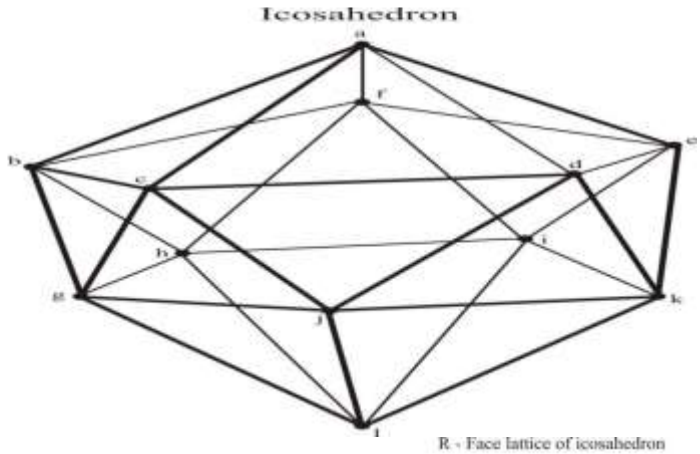
Proof

Since $Q \cong S(C_4)$ it is simple.

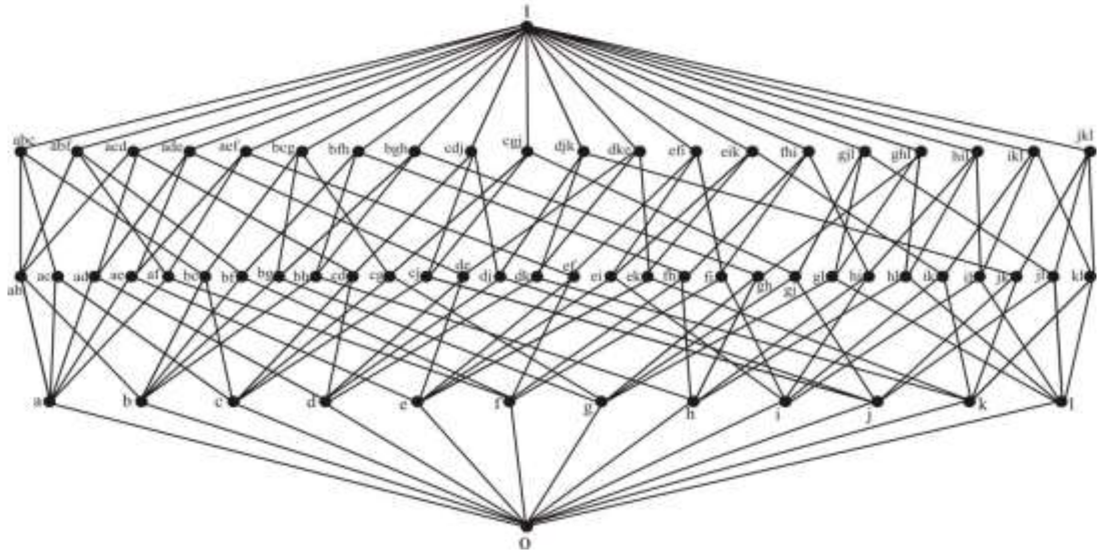
Theorem 3.2

The face lattice R of an icosahedron is simple.

Proof



R - Face lattice of icosahedron



Let θ be a congruence relation on R.

Since R is atomistic, there exists an atom x in R such that $(0,x) \in \theta$

Let y be the diametrically opposite vertex of x in the icosahedron, which is the constituent of R. Then there are 5 vertices connected with y with edges and faces which do not contain x . Let the vertices be y_1, y_2, y_3, y_4, y_5

Therefore in R, $x \vee y_1 = R$ and $x \vee y = R$

Similarly, $x \vee y_2 = R$, $x \vee y_3 = R$, $x \vee y_4 = R$, $x \vee y_5 = R$

Now $(0,x) \in \theta$ implies $(y,R) \in \theta$, $(y_1,R) \in \theta$, $(y_2,R) \in \theta$, $(y_3,R) \in \theta$, $(y_4,R) \in \theta$, $(y_5,R) \in \theta$ by taking join with $y, y_1, y_2, y_3, y_4, y_5$

Now take meet of two of these elements we get, $(0,R) \in \theta$.

Therefore, $\theta = R \times R$

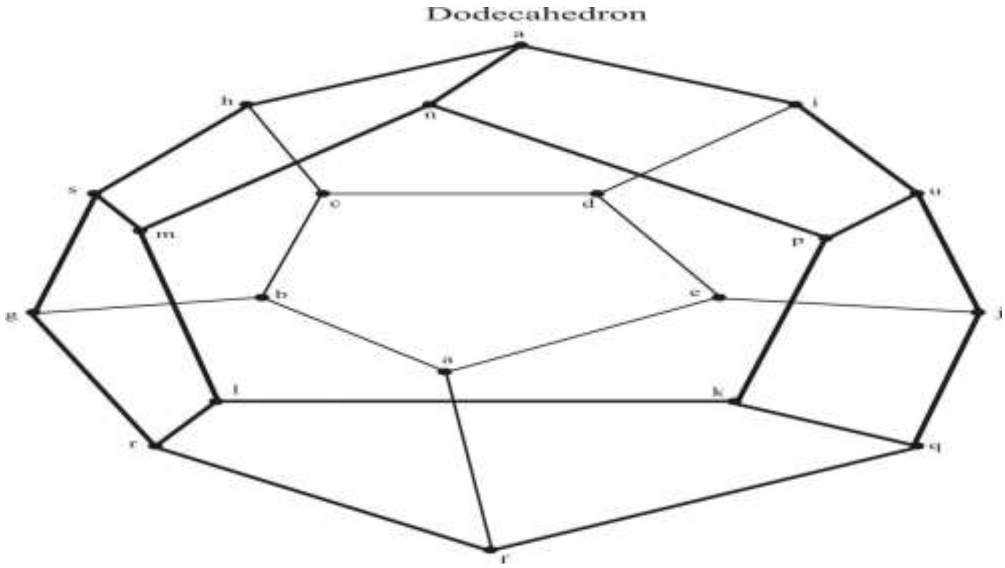
This is true for any atom a of R.

Therefore, we conclude that R is simple.

Theorem

The face lattice D of an dodecahedron is simple**.**

Proof



Let □ be a congruence relation on D.

Since D is atomistic, there exists an atom x in D such that □0,*x*□□□

Let y be a vertex in the face which is diametrically opposite to the face in which x is one of the vertices of Dodecahedron, which is the constituent of D.

Besides y, we have four more vertices in that face say, y_1, y_2, y_3 and y_4 .

Taking join of □*y*, *y*□*and* □*y*₁, *y*₁□*with*□0,*x*□□□ we get □*y*,*D*□,□*y*₁,*D*□□□. On taking meet we get □0,*D*□□□.

So, □□ *D*□*D*, the same argument is valid for all the other atoms.

Hence □□*D*□*D*

Hence, D contains no proper congruences.

Therefore, D is simple.

Theorem 3.2

The face lattice S of an octahedron is simple**.**

Proof

Let □ be a congruence relation on S.

Since S is atomistic, there exists an atom x in S such that □0,*x*□□□

Let y be the diametrically opposite vertex of x in the octahedron, which is the constituent of S. There is a face containing y which does not intersect of a face containing x. The face containing y contains two more vertices , say,y_1 and y_2. Taking join of (y_1,y_1) and (y_2,y_2) with (0,x) we get (y_1,D),(y_2,D) □□ Now take meet of these elements we get, □□,*S*□□□.

Therefore,□□*S*□*S*

This is true for any atom a of S.

Therefore, S has no proper congruences.

Therefore, we conclude that S is simple.

4. Eulerian property of the lattice of convex sublattices of some strongly uniform Eulerian Lattices

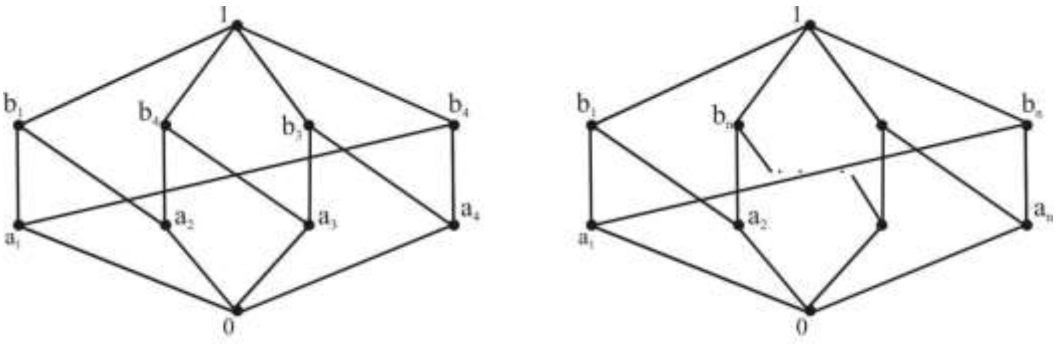
Lemma 4.1
A finite graded poset P is said to be Eulerian if and only if all intervals [x, y] of length $l \geq 1$ in P contain an equal number of elements of odd and even rank.

Lemma4.2
An Eulerian lattice of rank 3 which is strongly uniform is of the form

$$L = \bigcup_{i=1}^{r} \overline{C_{n_i}} \cup \{0,1\} \quad where \ \overline{C_{n_i}} = C_{n_i} \setminus \{0,1\}$$

Theorem 4.3
If L is strongly uniform Eulerian lattice of rank 3, then CS(L) is Eulerian. **Proof**
Since rank of L is 3, the rank of CS(L) is 4.
Let A_i be the number of elements of rank i in CS(L)
Then, $A_1 = 2(n_1+n_2+\ldots+n_r)+2$
$A_2 = n_1+n_2+\ldots+n_r + 2(n_1+n_2+\ldots+n_r)+ (n_1+n_2+\ldots+n_r)$
$A_3 = 2(n_1+n_2+\ldots+n_r)$
Hence, $A_1-A_2+A_3 = 2(n_1+n_2+\ldots+n_r)+2$
$-(n_1+n_2+\ldots+n_r) - 2(n_1+n_2+\ldots+n_r) - (n_1+n_2+\ldots+n_r)$
$+ 2(n_1+n_2+\ldots+n_r) = 2.$
Hence, CS(L) is Eulerian.

Corollary 4.4 CS(C_n) is Eulerian Proof



C_4 C_n

If we take one copy in the above theorem, then we get the result.

Remark

We have $CS(D_r(L)) = \bigcup_{j=1}^{r} \overline{CS(L_j)} \cup \{\phi, D_r(L)\}$

Theorem 4.5

$CS(D_r(L))$ is Eulerian if and only if $CS(L_j)$ is Eulerian for every j =1,2,…,r

Proof

Let $D_r(L)$ be of rank d+1,then each L_j is of rank d+1,rank of $CS(D_r(L))$ is d+2 and rank of each CS(L_j) is also d+2.

Let $CS(D_r(L))$ be Eulerian

Let $A_1, A_2, \ldots, A_{d+1}$ be number of elements of rank 1,2,…,d+1 of $CS(D_r(L))$.

Let $A_{j1}, A_{j2}, \ldots, A_{j(d+1)}$ be the number of elements of ranks 1,2,…,d+1 in CS(L_j).
Now, Let d be odd, then d+2 is also odd.

As $CS(D_r(L))$ is Eulerian, $A_1 - A_2 + \ldots + (-1)^{d-1} A_{d-1} = 0$
Suppose that for some j, CS(L_j) is not Eulerian.

Therefore, $A_{j1} - A_{j2} + \ldots + (-1)^{d-1} A_{j(d-1)} \neq 0$

But we have $A_1 = A_{11} + A_{21} + \ldots + A_{r1}$

$A_2 = A_{12} + A_{22} + \ldots + A_{r2}$

...etc

$A_{d-1} = A_{1(d-1)} + A_{2(d-1)} + \ldots + A_{r(d-1)}$

$$Now,\ A_1 - A_2 + \ldots + (-1)^{d-1} A_{d-1} = \left(A_{11} - A_{12} + A_{13} - \ldots + (-1)^{d-1} A_{1(d-1)}\right) + \left(A_{21} - A_{22} + A_{23} - \ldots + (-1)^{d-1} A_{2(d-1)}\right) + \ldots + \left(A_{j1} - A_{j2} + A_{j3} - \ldots + (-1)^{d-1} A_{j(d-1)}\right) + \ldots + \left(A_{r1} - A_{r2} + A_{r3} - \ldots + (-1)^{d-1} A_{r(d-1)}\right)$$

by (i)

Here, left side is 0 whereas the right side is non zero, which is a contradiction.
Therefore, our assumption is wrong.
Hence, each CS(L_j) is Eulerian.
The converse follows by a similar argument by view of (i).
When d is even , a similar argument holds.
Hence the theorem.

Lemma[8]

Let L be an Eulerian lattice of rank 4 which is strongly uniform and dual uniform. Then L is either B_4 or Q or R or their duals.

Theorem 4.3

If Q is the dual of the face lattice of a cube, then CS(Q) is Eulerian.

Proof.

Let Q be the dual of the face lattice of a cube
Now the rank of Q is 4
Let a_i be the number of elements of rank i in Q. then a_1= 6,a_2= 12 ,a_3=8
To prove that ,CS(Q) is Eulerian, we have to prove that the number of elements of even rank is equal to the number of elements of odd rank, in CS(Q).
Since rank of Q is 4, rank of CS(Q) is 5.
Let A_i be the number of elements of rank i in CS(Q), i=1,2,3,4
Therefore,

$A_1 =$ number of singleton sets of Q = 6 +12 +8+2 = 28

A_2 = number of edges of Q = 6 +24 +24 +8 = 62

A_3 = number of rank 2 convex sublattices of Q = 12 + 6 + 4 +12= 48

$A_4 =$ number of rank 3 convex sublattices of Q = 8 + 6 =14

Hence, $A_1 - A_2 + A_3 - A_4 = 28 - 62 + 48 - 14 = 0$
Therefore, CS(Q) is Eulerian.

Remark As Q = S(C_4), CS(Q) is Eulerian follows from the theorem 4.1 of [12].

Theorem 4.4

If R is the face lattice of an icosahedron, then CS(R) is Eulerian.

Proof

Let R be the face lattice of an icosahedron.
Now the rank of R is 4
Then the rank of CS(R) is 5
Let a_i be the number of elements of rank i in R. We have $a_1 = 12, a_2 = 30, a_3 = 20$. To prove that CS(R) is Eulerian, we have to prove that the number of elements of even rank is equal to number of elements of odd rank. Let A_i be the number of elements of rank i in CS(R)
Therefore,

A_1 = number of singleton sets of R = 12 +30 +20 +2= 64

A_2 = number of edges of R = 12 + (12 × 5) + (30 × 2) + 20
= 12 + 60 + 60 +20 = 152

A_3 = number of rank 2 convex sublattices of R = 30 +12 × 5 +30 × 1 = 120

A_4 = number of rank 3 convex sublattices of R = 12 + 20 = 32

Hence, $A_1 - A_2 + A_3 - A_4 = 64 - 152 + 120 - 32$
= 184-184 = 0.
Therefore, CS(R) is Eulerian.

Lemma[8]

A 4-simplicial strongly uniform Eulerian lattice of rank 5 with 8 atoms in which $[x,1] \cong Q$, for every atom x, is isomorphic to $((B_2 \backslash \{1\}) \times (Q \backslash \{1\})) \cup \{(1,1)\} = S(Q)$.

Theorem 4.5

Prove that CS[S(Q)] is Eulerian.

Proof

Since $S(Q) = ((B_2 \backslash \{1\}) \times (Q \backslash \{1\})) \cup \{(1,1)\}$,it is of rank 5,then the rank of CS[S(Q)] is 6

Now, $a_1 = 8, a_2 = 24, a_3 = 32, a_4 = 16$ where a_1, a_2, a_3, a_4 are the number of elements in S(Q) of ranks 1,2,3,4,respectively.

Let A_i be the number of elements of rank i in CS[S(Q)].

Then, $A_1 =$ number of singleton sets of $S(Q)$ = 8+24+32+16=82

$A_2 =$ number of edges of $S(Q)$ {0 at the bottom + rank 1 at the bottom+...}

$$= 8 + 8\binom{6}{1} + 24\binom{4}{1} + 32\binom{2}{1} + 16 = 8 + 8 \times 6 + 24 \times 4 + 32 \times 2 + 16$$

= 8 + 48+96+64+16 = 232

$A_3 =$ number of rank 2 convex sublattices of $S(Q)$
= 24 + [24+12 × 6]+[(6 × 4) × 2+(12 × 4)]+(2 × 12+8)
= 24+[24+12 × 6]+[(6 × 4) × 2+12 × 4]+[2 × 12+8]
= 24+96+96+32=246

$A_4 =$ number of rank 3 convex sublattices of $S(Q)$
= 32 + [2 × 8+ 8 × 6]+[2 × 6+12]
= 32 + 64 + 24 =120

$A_5 =$ number of rank 4 convex sublattices of $S(Q)$
= (2 × 8)+(2 × 1+6)
= 16+2+6=24

$A_1 - A_2 + A_3 - A_4 + A_5$ = 82 – 232 + 246 – 120 + 24 = 2.

Hence, $CS[S(Q)]$ is Eulerian.

Theorem

Prove that CS[S(R)] is Eulerian.

Proof

Since $S(R) \cong [B_2 \setminus \{1\}] \times [R \setminus \{1\}] \oplus \{1,1\}$, it is of rank 5,then the rank of CS[S(R)] is 6

Now,$a_1 = 14, a_2 = 54, a_3 = 80, a_4 = 40$ where a_1, a_2, a_3, a_4 are the number of elements in S(R) of ranks 1,2,3,4,respectively.

We observe that if x is any atom in an extreme copy of S(R), then $[x,1] \cong C_5$ and if it is in the middle copy of S(R), then $[x,1] \cong S(C_5)$.

We also note that if y is an element of rank 2 in the middle copy of S(R), then $[y,1] \cong S(B_2) \times C_4$ Let A_i be the number of elements of rank i in CS[S(R)].

Then, $A_1 =$ number of singleton sets of S(R) = 16+54+80+40=190.

A_2 = Number of edges of S(R) = number of edges containing zero+ number of edges containing an atom + number of edges containing rank 2 elements + number of edges containing rank 3 elements + number of edges containing rank 4 elements at the bottom

= [12 + 2] +[2 ×12 +12×7] + [(12 × 5) × 2+ 30× 4]+[(30× 2) × 2+20× 2]+[20+20] = 14+24+84+120+120+120+60+20 = 562.

A_3 = number of rank 2 convex sublattices of S(R).

= 54+[2×30+15 ×12]+[(12×5) × 2+30× 4]+[(30× 1) × 2]+20× 1

= 54+240+240+60+20=614

A_4 = number of rank 3 convex sublattices of S(R).

= 80+2×20+12×10+2×12+30

= 80+40+120+24+30=294

A_5 = number of rank 4 convex sublattices of S(R).

= 40+14=54

$A_1-A_2+A_3-A_4+A_5$ = 190-562+614-294+54=858-856=2 Hence,$CS[S(R)]$ is Eulerian.

5.Lattice of convex sublattices of a dual simplicial Eulerian lattice Theorem 5.1

Lattice of convex sublattices of any dual simplicial Eulerian lattice is Eulerian.

Proof.

Let L be a dual simplicial Eulerian lattice of rank d+1 with a_i , number of elements of ranks i=1,2...,d Then the rank of CS(L) is d + 2.

Let d be even

Therefore, $a_1 - a_2 + \ldots + (-1)^{d-1} a_d = 0$

Claim: CS(L) is Eulerian

Let A_i be the number of elements of rank i in CS(L)

Now,

$A_1 = a_1 + a_2 + \ldots + a_d + 2$

A_2 = number of edges with {0 at the bottom + an atom at the bottom + a rank 2 element at the bottom + ... + a rank d elements at the bottom}

= $a_1 + (a_1 d + a_2(d-1) + \ldots + a_d)$

A_3 = number of rank 2 convex sublattices with {0 at the bottom + an atom at the bottom + ... + a rank (d-1) element at the bottom}

= $a_2 + \binom{d}{2} a_1 + \binom{d-1}{2} a_2 + \ldots + a_{d-1}$

A_4 = number of rank 3 convex sublattices with {0 at the bottom + an atom at the bottom + ... + a rank (d-2) element at the bottom}

$(d) \quad (d-1)$

= a_3 □□ □ 3□ □ □a_1 □□□ □ 3 □ □□a_2 □…□$a_{d□2}$

□ And so on .

A_d = number of rank (d-1) convex sublattices with {0 at the bottom + an atom at the bottom + a rank 2 element at the bottom}

□ d □

A_d □$a_{d□1}$ □□□ □a

□d □1□□ $_1$ □a_2

$A_{d□1}$ = number of rank d convex sublattices

= number of convex sublattices of rank d{containing 0 at the bottom + containing an atom at the bottom} $A_{d□1}$ = a_1 □a_d

Hence A_1 □A_2 □A_3 □A_4 □…□A_d □$A_{d□1}$

= □a_1 □□$a_1 d$□a_2□d□1□□…□a_d □□□ □□□□a_2 □ □□ □□d2□□□ □ a_1 □ □ □□ □ 2d □1 □□ □□a_2

□…□$a_{d□1}$□□□□□

□ □d□ □d □1□ □

□□a_3 □□□3 □□□a_1 □□□□ 3□□□a_2 □…□$a_{d□2}$ □□□□

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2□a_2 □a_4 □…□a_d □□ 2□$a_1 d$ □a_2□d □1□□…□a_d □

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=

a_1□□□2 □□□C_2 □a_2□□□ 2 □□□C_2 □…□$a_{d□1}$ □a_1□□□3 □□□□a_2□□□3 □□□□…

□ d □

□$a_{d□2}$ □…□a_1□□□d □1□□□□a_2 □a_1.

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□a_2□□□□□ 1 □□□□□□□ 2 □□□□□□□ 3 □□□□…□□□□d □1□□□□□□…□$a_{d□1}$□2 □1□□a_d .

□

2□a_2 □a_4 □…□a_d □□ 2□a_1 □1□□1□1□^{d} □□a_2□1□□1□1□$^{d□1}$□

□a_3□1□□1□1□$^{d□2}$□□…□$a_{d□1}$ □a_d

= 2□a_2 □a_4 □…□a_d□□ 2□a_1 □a_2 □a_3 □…□$a_{d□1}$ □a_d

□ □a_1 □a_2 □a_3 □a_4 □…□$a_{d□1}$ □a_d □ 2

= 0 + 2 = 2. If d is odd, then

A_1 □A_2 □A_3 □…□A_d □$A_{d□1}$

= a_1 □a_2 □…□a_d □ 2□□a_1 □$a_1 d$□a_2 □…□a_d □□ □□□a_2 □a_1□□□d2□□□□□a_2□□□□d2□1□□□□□…□$a_{d□1}$□□□□

□□□□a_3 □a_1□□□3 □□□C_3 □a_2□□□ 3□□□□…□$a_{d□2}$ □□□ +…+ □□□$a_{d□1}$ □a_1□□□d □1□□□□a_2 □□□□□a_1 □a_d □

= 2□a_2 □a_4 □…□$a_{d□1}$□□ 2 + a_1□□□□d □□□□d2□□□□□□□□□3d□□□□□…□□□□□dd □1□□□□□1□□□□

+ a_2□□□□□d □1□□□□d2□1□□□□□□□□d2□1□□□□□□□□d3□1□□□□□…□1□□□□

+ a_3□□□□d □ 2□□□□ 3□□□□…□1□□□□…□a_d □□1□

= 2□a_2 □a_4 □…□$a_{d□1}$□□ 2 + a_1□□1□1□^{d}□^{1}□□a_2□□1□1□$^{d□1}$□^{1}□

+ a_3□□1□1□$^{d□2}$□1□□…□a_d□□1□

= 2□a_2 □a_4 □…□$a_{d□1}$□□ 2□a_1 □a_2 □…□a_d

= □a_1 □a_2 □a_3 □a_4 □a_5 □…□$a_{d□1}$ □a_d

= 0

Hence, the Lattice of all convex sublattices of a dual simplicial Eulerian lattice is Eulerian.

Conclusion

We have proved above that the lattice of convex sublattices of some known simple Eulerian lattices are Eulerian. Also we have investigated the truthfulness of the Eulerian property for strongly uniform and dual uniform Eulerian lattices upto rank 5 only. The study on the strongly uniform and dual uniform Eulerian lattices of ranks > 5 is also possible, but it looks difficult.

References

1. 1.Gratzer,G., General lattice theory, Birkhauser Verlag, Basel,1978.
2. 2.Koh, K.M.,On the lattice of convex sublattices of a lattice, Nantha Math., 6,18–37,1972.
3. 3.Paffenholz A.,Construction for posets, Lattices and polytopes, Doctoral Dissertation, School of Mathematics and Natural Sciences ,Technical University of Berlin,2005.
4. 4.Randolph Stonsifer .J,Modularly complemented geometric lattices, Discrete Math.32,1980,85-88.
5. V.K.Santhi, Topics in Commutative Algebra,Ph.D.Thesis(Madurai Kamaraj University),1992.
6. 6.Stanley R.P., Enumerative Combinatorics, Vol. 1, Wordsworth and Brooks/Cole, Monterey, CA, 1986.
7. 7.K.E.Usha Nirmala Kumari, A New approach to some problems on Lattices, Ph.D thesis, Bharathidasan University, August 2022.
8. 8.Vethamanickam A., Topics in Universal Algebra, Ph.D thesis, Madurai Kamaraj University,1994.
9. 9.Vethamanickam .A and Aaswin.J, The lattice of convex sublattice of $S(S(B_n))$, Eur. Chem. Bull.,12(Special issue 6),8097-8106, 2023.
10. Vethamanickam .A and Usha Nirmala Kumari.K.E, The lattice of convex sublattice of CS□S□B_n□C_m□□ (communicated).
11. 11.Vethamanickam. A and Sheeba Merlin.G, Simplicity of the lattice of convex sublattices, International Journal of Mechanical Engineering and Technology (IJMET) Volume 9, Issue 11, pp. 383–387, October 2018.
12. 12.Vethamanickam. A and Sheeba Merlin. G, On the lattice of convex sublattices of $S(B_n)$ and $S(C_n)$, European Journal of Pure and Applied Mathematics,Vol.10,No..4, 916-928, 2017.

13. 13.Vethamanickam A. and Subbarayan R., Some Simple Extensions of Eulerian Lattices, Acta Mathematica Universitatis Comeninae New Series 79(1),47-54,2010.
14. 14.Vethamanickam,A . , and Subbarayan,R . , On the lattice of convex sublattices, Elixir Dis . Math . 50,10471–10474, 2012.