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Research Article

On Distance Related Spectrum Of Nil-Graph Of Ideals Of Commutative Rings

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ARTICLE INFO **ABSTRACT**

Let R be a commutative ring with identity and $Nil(R)$ be the ideal of all nilpotent elements of R. Let $\mathrm{I}(R) = \{I: I$ is a non-trivial ideal of R and there exists a non-trivial ideal J such that $IJ \subseteq Nil(R)$.$\}$ The nil-graph of ideals of R is defined as the graph $\mathrm{AG}_N(R)$ whose vertex set is the set $\mathrm{I}(R)$ and two distinct vertices I and J are adjacent if and only if $IJ \subseteq Nil(R)$. In this paper, we determine distance signless Laplacian and distance Laplacian spectrum of $\mathrm{AG}_N(R)$ when R is reduced.

Keywords: Distance signless Laplacian matrix, Distance Laplacian matrix, Nil-graph

Subject Classification: 05C25, 13A15, 05C12,15A18

1 Introduction

G is a graph with vertex set $V(G)$ and edge set $E(G)$. The adjacency matrix $A(G) = (a_{ij})$ of G is a square matrix of order n, whose (i,j)-entry is 1, if v_i and v_j are adjacent and is 0, otherwise. Let $Deg(G) = diag(d_1, d_2, \cdots, d_n)$ be the diagonal matrix, where $d_i = d_G(v_i)$ are the degrees of the vertices of G. The matrix $L(G) = Deg(G) - A(G)$ and $Q(G) = Deg(G) + A(G)$ are called the Laplacian and the signless Laplacian matrices and their eigenvalues with multiplicities are known as the Laplacian spectrum and the signless Laplacian spectrum of the graph G.

In a graph G , the distance between any two vertices $u, v \in V(G)$, denoted by $d(u, v)$, is defined as the length of a shortest path between u and v. The diameter of G is the maximum distance between any two vertices of G. The distance matrix of G, denoted by $D(G)$, is defined as $D(G) = [d(u, v)]$, where $u, v \in V(G)$. The transmission degree $Tr_G(v)$ of a vertex v is defined to be the sum of the distances from v to all other vertices in G, that is, $Tr_G(v) = \sum_{u \in V(G)} d(u, v)$. If $Tr_G(v_i)$ (or simply Tr_i) is the transmission degree of the vertex $v_i \in V(G)$, the sequence $\{Tr_1, Tr_2, \cdots, Tr_n\}$ is called the transmission degree sequence of the graph G.

Let $Tr(G) = diag[Tr_1, Tr_2, \cdots, Tr_n]$ be the diagonal matrix of vertex transmissions of G. Aouchiche and Hansen [1] introduced the Laplacian and the signless Laplacian for the distance matrix of a connected graph. The matrix $D^L(G) = Tr(G) - D(G)$ is called the distance Laplacian matrix of G. The matrix $D^L(G) = Tr(G) - D(G)$ is real symmetric and positive semi-definite, so we order the distance Laplacian eigenvalues as $\partial_1^L(G) \geq \cdots \geq \partial_{n-1}^L(G) \geq \partial_n(G) = 0$, where $\partial_1^Q(G)$ is called the distance Laplacian spectral radius of G. The matrix $D^Q(G) = Tr(G) + D(G)$ is called the distance signless Laplacian matrix of G. The matrix $D^Q(G) = Tr(G) + D(G)$ is real symmetric and positive definite for $n \geq 3$, so its eigenvalues can be arranged as $\rho_1^Q(G) \geq \cdots \geq \rho_{n-1}^Q(G) \geq \rho_n^Q(G)$, where $\rho_1^Q(G)$ is called the distance signless Laplacian spectral radius of G. For detailed notion of spectrum of graphs one can refer [4, 5, 6, 9].

A commutative ring R is called a local ring if it has a unique maximal ideal. Throughout this paper R denotes a commutative Artinian nonlocal ring with identity and which is not an integral domain. We call an ideal I of R, an annihilating-ideal if there exists a non-zero ideal J of R such that $IJ = (0)$. Let $A^*(R)$ be the set of non-zero annihilating ideals of R. Behboodi and Rakeei [2, 3] have introduced and investigated the annihilating-ideal graph of a commutative ring. The annihilating-ideal graph of R is defined as the graph $\mathbb{AG}(R)$ with the vertex set $A^*(R)$ and two distinct vertices I and J are adjacent if and only if $IJ = (0)$. In [7], Shaveisi et al. extended this notion of the annihilating-ideal graph as the nil-graph of ideals of R. Let $Nil(R)$ be the ideal of all nilpotent elements of R and $\mathbb{I}(R) = \{I : I$ is a non-trivial ideal of R and there exists a non-trivial ideal J such that $IJ \subseteq Nil(R)\}$. The nil-graph of ideals of R is defined as the graph $\mathbb{AG}_N(R)$ whose vertex set is the set $\mathbb{I}(R)$ and two distinct vertices I and J are adjacent if and only if $IJ \subseteq Nil(R)$. Obviously the notion of nil-graph of ideals is different from the notion of annihilating-ideal graph and it is easy to see that $\mathbb{AG}(R)$ is a subgraph of $\mathbb{AG}_N(R)$.

Let $G(V,E)$ be a graph of order n and $G_i(V_i,E_i)$ be graphs of order n_i, where $i = 1,\cdots,n$. The joined union $G[G_1,\cdots,G_n]$ is the graph $G^*(W,F)$ with $W = \bigcup_{i=1}^{n} V_i$ and $F = \bigcup_{i=1}^{n} E_i \cup \bigcup_{\{v_i,v_j\}\in E} V_i \times V_j$.

The rest of the paper is organized as follows. In section 2, we view $\mathbb{AG}_N(R)$ as joined union of graphs when R is reduced. In section 3, we view $\mathbb{AG}_N(R)$ as joined union of graphs using canonical representation when R is reduced. In section 4, we determine transmission degree of each vertex of $\mathbb{AG}_N(R)$ when R is reduced and we state two Lemmas proved in [5, 6] which are used in the subsequent sections. In Section 5, 6, we investigate the distance signless laplacian and distance Laplacian spectrum of $\mathbb{AG}_N(R)$ respectively when R is reduced. Also we note that if R is Artinian reduced then R is isomorphic to finite direct product of fields and $Nil(R) = <0>$. We have used computational software, Wolfram Mathematica for computing approximate eigenvalues. For any set A, we denote the complement of A by A'. We denote transmission degree of a vertex I_A in $\mathbb{AG}_N(R)$ by $Tr(I_A)$.

2 Nil-graph as joined union of graphs

In this section the nil-graph of ideals of a commutative Artinian ring R is viewed as joined union of suitable choices of graphs when R is reduced.

Assume that $R = F_1 \times \cdots \times F_n$ where each F_i is a field and $n \geq 2$. We define an equivalence relation $\sim$ on $\mathbb{I}(R)$ as follows. For $I, J \in \mathbb{I}(R)$, define $I \sim J$ if and only if $N(I) = N(J)$ in $\mathbb{AG}_N(R)$, where $N(I) = \{K \in \mathbb{I}(R) : K \neq I, IK = <0>\}$.

Let $C_1,\ldots,C_t$ be the equivalence classes of this relation with respective representatives $\mathbf{J}_1,\cdots,\mathbf{J}_t \in \mathbb{I}(R)$ and $C_i = \{I \in \mathbb{I}(R) : N(I) = N(\mathbf{J}_i)\}$. Then $\mathbb{I}(R) = \cup_{i=1}^{t} C_i$. First we see some properties of $C_i's$.

Lemma 2.1. [8]

(i) For $i, j \in \{1, \cdots, t\}$ and $i \neq j$, a vertex of C_i is adjacent to a vertex of C_j in $\mathbb{AG}_N(R)$ if and only if $\mathbf{J}_i\mathbf{J}_j =< 0 >$.

(ii) $|C_i| = 1$ for all $i = 1, \cdots, t$.

We define $H(R)$ (or simply H) as the simple graph whose vertices are the representatives $\mathbf{J}_1, \cdots, \mathbf{J}_t$ in which two distinct vertices are adjacent if and only if $\mathbf{J}_i.\mathbf{J}_k =< 0 >$.

Lemma 2.2. [8] The graph H is connected.

Proposition 2.3. [8]

Let G_i be the subgraph induced by the set C_i for all $i = 1, \cdots, t$ in $\mathbb{AG}_N(R)$. Then $\mathbb{AG}_N(R) = H[G_1, G_2, \cdots, G_t]$.

3 Partition for nil-graph

Let R be a commutative reduced ring and $\mathcal{P}_n^* = \{A : A \subsetneq [n],\ A \neq \emptyset\}$, where $[n] = \{1, \cdots, n\}$. For $A \in \mathcal{P}_n^*$, we define the characteristic vector of A to be the ideal $I_A = I_1 \times \cdots \times I_n$ in R satisfying $I_l = F_l$ if $l \notin A$ and $I_l =< 0 >$ otherwise. Also for $A \in \mathcal{P}_n^*$, we define the set

$$C_A = \{I_1 \times \cdots \times I_n \text{ is an ideal in } R :\ I_i = F_i \text{ if and only if } i \notin A\}.$$

From these notions and the results discussed in Section 2, we have the following two lemmas. The lemmas explain about the non-zero proper ideals of R and the cardinalities of the equivalence classes C_A of R.

Lemma 3.1. [8] Consider the equivalence relation $\sim$ on $\mathbb{I}(R)$ given by $I \sim J$ if and only if $N(I) = N(J)$ in $\mathbb{AG}_N(R)$.

(i) Let $I_1 \times \cdots \times I_n$ and $J_1 \times \cdots \times J_n$ be two elements of $\mathbb{I}(R)$, Then $N(I) = N(J)$ if and only if Supp $I := \{\ i :\ 1 \leq i \leq n, I_i = R_i\} = \{\ j :\ 1 \leq j \leq n, J_j = R_j, \} :=$ Supp J.

(ii) The equivalence classes of the equivalence relation $\sim$ are precisely the sets $C_A = C_i$ for $A \in \mathcal{P}_n^*$. In particular

$$\mathbb{I}(R) = \bigcup_{A \in \mathcal{P}_n^*} C_A.$$

(iii) The characteristic vector $I_A = \mathbf{J}_i$ of set $A \in \mathcal{P}_n^*$ can be taken as the canonical representative of the class C_A. Therefore $2^n - 2$ distinct equivalence classes in $\mathbb{I}(R)$ for the relation $\sim$.

(iv) $|C_A| = 1$ for all $A \subsetneq [n]$.

In the following lemma, we calculate the distance between two vertices of H.

The vertex set of H is $V(H) = \{I_A : A \in \mathcal{P}_n^*\}$ and the vertices I_A and I_B are adjacent in H if and only if $I_AI_B \subseteq Nil(R)$, if and only if $A \cup B = [n]$.

Lemma 3.2. [8] Let I_A and I_B be two distinct vertices of H, then

$$d_H(I_A, I_B) = \begin{cases} 1 & \text{if } A \cup B = [n] \\ 2 & \text{if } A \cap B \neq \emptyset \text{ and } A \cup B \subsetneq [n] \\ 3 & \text{if } A \cap B = \emptyset \text{ and } A \cup B \subsetneq [n] \end{cases}$$

4 Distance Laplacian and distance signless Laplacian spectrum of graphs

In this section we find the transmission degree of a vertex $I_A \in V(\mathbb{AG}_N(R))$. Also we see two lemmas proved in [5, 6] which are used in the subsequent sections.

If $R = F_1 \times F_2 \times \cdots \times F_n$, we find the transmission degree $Tr(I_A)$ for each I_A in $V(\mathbb{AG}_N(R))$. If $I_A \in V(\mathbb{AG}_N(R))$, we define $S_{A,1} = \{B \in P_n^* : A \cup B = [n]\}$,

$S_{A,2} = \{B \in P_n^* : A \cap B \neq \emptyset \text{ and } A \cup B \subsetneq [n]\}$,

$S_{A,3} = \{B \in P_n^* : A \cap B = \emptyset \text{ and } A \cup B \subsetneq [n]\}$.

We have $|S_{A,1}| = \binom{|A|}{0} + \binom{|A|}{1} + \cdots + \binom{|A|}{|A|-1} = 2^{|A|} - 1$, $|S_{A,3}| = 2^{|A'|} - 2$, $|S_{A,2}| = (2^n - 3) - (|S_{A,1}| - |S_{A,3}|)$.

Theorem 4.1. If $R = F_1 \times F_2 \times \cdots \times F_n$ and $I_A \in V(\mathbb{AG}_N(R))$, then $Tr(I_A) = 1(|S_{A,1}|) + 2(|S_{A,2}|) + 3(|S_{A,3}|)$.

Proof. $Tr(I_A) = \sum_{I_B \in V(\mathbb{AG}_N(R))} d(I_A, I_B)$.

Since diameter of $\mathbb{AG}_N(R) \leq 3$, we have, $Tr(I_A) = 1(|S_{A,1}|) + 2(|S_{A,2}|) + 3(|S_{A,3}|)$. □

Lemma 4.2. [5] Let G be a graph of order n having vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$. Let G_i be r_i regular graphs of order n_i having adjacency eigenvalues $\lambda_{i1} = r_i \geq \lambda_{i2} \geq \cdots \geq \lambda_{in_i}$, where $i = 1, 2, \cdots, n$. The distance signless Laplacian spectrum of the joined union graph $G[G_1, G_2, \cdots, G_n]$ of order $\sum_{i=1}^{n} n_i$ consists of the eigen values $2n_i + n_i' - r_i - \lambda_{ik} - 4$ for $i = 1, \cdots, n$ and $k = 2, 3, \cdots, n_i$, where $n_i' = \sum_{k=1, k\neq i}^{n} n_k d_G(v_i, v_k)$. The remaining n eigenvalues are given by the equitable quotient matrix

$$Q = \begin{pmatrix} 4n_1 + n_1' - 2r_1 - 4 & n_2 d_G(v_1, v_2) & \cdots & n_n d_G(v_1, v_n) \\ n_1 d_G(v_2, v_1) & 4n_2 + n_2' - 2r_2 - 4 & \cdots & n_n d_G(v_2, v_n) \\ \vdots & \vdots & \vdots & \\ n_1 d_G(v_n, v_1) & n_2 d_G(v_n, v_2) & \cdots & 4n_n + n_n' - 2r_n - 4 \end{pmatrix}.$$

Lemma 4.3. [6] Let G be a graph of order n having vertex set $V(G) = \{v_1, v_2, \cdots v_n\}$. Let G_i be a graph of order m_i with Laplacian eigenvalues $\mu_{i1} \geq \mu_{i2} \geq \cdots \geq \mu_{im_i}$, where $i = 1, 2, \cdots, n$. The distance Laplacian spectrum of the joined union $G[G_1, G_2, \cdots, G_n]$ consists of the eigenvalues $2m_i - \mu_{ik} + \alpha_i$ for $i = 1, \cdots, n$ and $k = 1, 2, 3, \cdots, m_i - 1$, where $\alpha_i = \sum_{k=1, k\neq i}^{n} m_k d_G(v_i, v_k)$. The remaining n eigenvalues are given by the matrix

$$M = \begin{pmatrix} \alpha_1 & -m_2 d_G(v_1, v_2) & \cdots & -m_n d_G(v_1, v_n) \\ -m_1 d_G(v_2, v_1) & \alpha_2 & \cdots & -m_n d_G(v_2, v_n) \\ \vdots & \vdots & \vdots & \\ -m_1 d_G(v_n, v_1) & -m_2 d_G(v_n, v_2) & \cdots & \alpha_n \end{pmatrix}.$$

5 Distance signless Laplacian spectrum of nil-graph

In this section we find distance signless Laplacian Spectrum of $\mathbb{AG}_N(R)$ when $R = F_1 \times F_2 \times \cdots \times F_n$. In particular, we determine the distance signless Laplacian spectrum of $\mathbb{AG}_N(F_1 \times F_2)$, $\mathbb{AG}_N(F_1 \times F_2 \times F_3)$, $\mathbb{AG}_N(F_1 \times F_2 \times F_3 \times F_4)$,

Lemma 5.1. If $R = F_1 \times F_2 \times \cdots \times F_n$ then the distance signless Laplacian spectrum of $\mathbb{AG}_N(R)$ are the zeros of the characteristic polynomial of the equitable quotient matrix

$$Q = \begin{pmatrix} Tr(I_{A_1}) & d(I_{A_1}, I_{A_2}) & \cdots & d(I_{A_1}, I_{A_{2^n-2}}) \\ d(I_{A_2}, I_{A_1}) & Tr(I_{A_2}) & \cdots & d(I_{A_2}, I_{A_{2^n-2}}) \\ \vdots & \vdots & \vdots & \\ d(I_{A_{2^n-2}}, I_{A_1}) & d(I_{A_{2^n-2}}, I_{A_2}) & \cdots & Tr(I_{A_{2^n-2}}) \end{pmatrix}.$$

Proof. If $R = F_1 \times F_2 \times \cdots \times F_n$ then $\mathcal{P}_n^* = \{A : A \subsetneq [n], A \neq \emptyset\}$, and $V(\mathbb{AG}_N(R)) = \{I_A : A \in \mathcal{P}_n^*\}$. We have $2^n - 2$ elements in $V(\mathbb{AG}_N(R))$. Let $\mathcal{P}_n^* = \{A_1, A_2, \cdots, A_{2^n-2}\}$. Each G_i has exactly one element and so the adjacency eigen value of each G_i is 0 for $i = 1, 2, \cdots, 2^n - 2$. Now $n_i' = \sum_{k=1, k\neq i}^{2^n-2} d(I_{A_i}, I_{A_k}) = Tr(I_{A_i})$. By Lemma 4.2, we have

$$Q = \begin{pmatrix} Tr(I_{A_1}) & d(I_{A_1}, I_{A_2}) & \cdots & d(I_{A_1}, I_{A_{2^n-2}}) \\ d(I_{A_2}, I_{A_1}) & Tr(I_{A_2}) & \cdots & d(I_{A_2}, I_{A_{2^n-2}}) \\ \vdots & \vdots & \vdots & \\ d(I_{A_{2^n-2}}, I_{A_1}) & d(I_{A_{2^n-2}}, I_{A_2}) & \cdots & Tr(I_{A_{2^n-2}}) \end{pmatrix}.$$

Lemma 5.2. If $R = F_1 \times F_2$ then the distance signless Laplacian spectrum of $\mathbb{AG}_N(R)$ are $0, 2$.

Proof. We have $\mathcal{P}_2^* = \{\{1\}, \{2\}\}$, $Tr(I_{\{1\}}) = 1, Tr(I_{\{2\}}) = 1$ and $d(I_{\{1\}}, I_{\{2\}}) = 1$, By Lemma 5.1, $Q = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and the eigenvalues of Q are $0, 2$.

Lemma 5.3. If $R = F_1 \times F_2 \times F_3$ then the distance signless Laplacian spectrum of $\mathbb{AG}_N(R)$ are $7 + \sqrt{2}^{[2]}, 7 - \sqrt{2}^{[2]}, 13 + \sqrt{41}, 13 - \sqrt{41}$.

Proof. We have $\mathcal{P}_3^* = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$, $|S_{\{1\},1}| = |S_{\{2\},1}| = |S_{\{3\},1}| = 1, |S_{\{1\},3}| = |S_{\{2\},3}| = |S_{\{3\},3}| = 2^2 - 2 = 2, |S_{\{1\},2}| = |S_{\{2\},2}| = |S_{\{3\},2}| = 5 - 1 - 2 = 2$
$|S_{\{1,2\},1}| = |S_{\{1,3\},1}| = |S_{\{2,3\},1}| = 2^2 - 1 = 3$,
$|S_{\{1,2\},3}| = |S_{\{1,3\},3}| = |S_{\{2,3\},3}| = 2^1 - 2 = 0$,
$|S_{\{1,2\},2}| = |S_{\{1,3\},2}| = |S_{\{2,3\},2}| = 5 - 3 = 2$.
Now $Tr(I_{\{1\}}) = Tr(I_{\{2\}}) = Tr(I_{\{3\}}) = 1(1) + 2(2) + 3(2) = 11$
$Tr(I_{\{1,2\}}) = Tr(I_{\{1,3\}}) = Tr(I_{\{2,3\}}) = 1(3) + 2(2) = 7$.

$$Q = \begin{pmatrix} 11 & 3 & 3 & 2 & 2 & 1 \\ 3 & 11 & 3 & 2 & 1 & 2 \\ 3 & 3 & 11 & 1 & 2 & 2 \\ 2 & 2 & 1 & 7 & 1 & 1 \\ 2 & 1 & 2 & 1 & 7 & 1 \\ 1 & 2 & 2 & 1 & 1 & 7 \end{pmatrix}$$

and the eigenvalues of Q are $7 + \sqrt{2}^{[2]}, 7 - \sqrt{2}^{[2]}, 13 + \sqrt{41}, 13 - \sqrt{41}$.

□

Lemma 5.4. If $R = F_1 \times F_2 \times F_3 \times F_4$ then the distance signless Laplacian spectrum of $\mathbb{AG}_N(R)$ are $52.9, 23+\sqrt{30}^{[3]}, 24.7, 24^{[3]}, 22^{[2]}, 18.4, 23-\sqrt{30}^{[3]}$.

Proof. $\mathcal{P}_4^* = \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\{1,2,3\},$ $\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$.

If $|A| = 1$ then $|S_{A,1}| = 2^1 - 1 = 1, |S_{A,3}| = 2^3 - 2 = 6, |S_{A,2}| = 2^4 - 3 - 1 - 6 = 6$ and $Tr(I_A) = 1(1) + 2(6) + 3(6) = 31$.

If $|A| = 2$ then $|S_{A,1}| = 2^2 - 1 = 3, |S_{A,3}| = 2^2 - 2 = 2, |S_{A,2}| = 2^4 - 3 - 3 - 2 = 8$ and $Tr(I_A) = 1(3) + 2(8) + 3(2) = 25$.

If $|A| = 3$ then $|S_{A,1}| = 2^3 - 1 = 7, |S_{A,3}| = 2^1 - 2 = 0, |S_{A,2}| = 2^4 - 3 - 7 = 6$ and $Tr(I_A) = 1(7) + 2(6) = 19$.

$$Q = \begin{pmatrix}
31 & 3 & 3 & 3 & 2 & 2 & 2 & 3 & 3 & 3 & 2 & 2 & 2 & 1 \\
3 & 31 & 3 & 3 & 2 & 3 & 3 & 2 & 2 & 3 & 2 & 2 & 1 & 2 \\
3 & 3 & 31 & 3 & 3 & 2 & 3 & 2 & 3 & 2 & 2 & 1 & 2 & 2 \\
3 & 3 & 3 & 31 & 3 & 3 & 2 & 3 & 2 & 2 & 1 & 2 & 2 & 2 \\
2 & 2 & 3 & 3 & 25 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 1 & 1 \\
2 & 3 & 2 & 3 & 2 & 25 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 1 \\
2 & 3 & 3 & 2 & 2 & 2 & 25 & 1 & 2 & 2 & 1 & 2 & 2 & 1 \\
3 & 2 & 2 & 3 & 2 & 2 & 1 & 25 & 2 & 2 & 2 & 1 & 1 & 2 \\
3 & 2 & 3 & 2 & 2 & 1 & 2 & 2 & 25 & 2 & 1 & 2 & 1 & 2 \\
3 & 3 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 25 & 1 & 1 & 2 & 2 \\
2 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 19 & 1 & 1 & 1 \\
2 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 1 & 1 & 19 & 1 & 1 \\
2 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 1 & 2 & 1 & 1 & 19 & 1 \\
1 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 19
\end{pmatrix}$$

and the eigenvalues of Q are $52.9, 23+\sqrt{30}^{[3]}, 24.7, 24^{[3]}, 22^{[2]}, 18.4, 23-\sqrt{30}^{[3]}$.

□

6 Distance Laplacian spectrum of nil-graph

In this section we find distance Laplacian spectrum of $\mathbb{AG}_N(R)$ when $R = F_1 \times F_2 \times \cdots \times F_n$. In particular, we determine the distance Laplacian Spectrum of $\mathbb{AG}_N(F_1 \times F_2)$, $\mathbb{AG}_N(F_1 \times F_2 \times F_3)$, $\mathbb{AG}_N(F_1 \times F_2 \times F_3 \times F_4)$,

Lemma 6.1. If $R = F_1 \times F_2 \times \cdots \times F_n$ then the distance Laplacian spectrum of $\mathbb{AG}_N(R)$ are the zeros of the characteristic polynomial of the matrix

$$M = \begin{pmatrix}
Tr(I_{A_1}) & -d(I_{A_1}, I_{A_2}) & \cdots & -d(I_{A_1}, I_{A_{2^n-2}}) \\
-d(I_{A_2}, I_{A_1}) & Tr(I_{A_2}) & \cdots & -d(I_{A_2}, I_{A_{2^n-2}}) \\
\vdots & \vdots & \vdots & \\
-d(I_{A_{2^n-2}}, I_{A_1}) & -d(I_{A_{2^n-2}}, I_{A_2}) & \cdots & Tr(I_{A_{2^n-2}})
\end{pmatrix}.$$

Proof. If $R = F_1 \times F_2 \times \cdots \times F_n$ then $\mathcal{P}_n^* = \{A : A \subsetneq [n],\ A \neq \emptyset\}$, and $V(\mathbb{AG}_N(R)) = \{I_A : A \in \mathcal{P}_n^*\}$. We have $2^n - 2$ elements in $V(\mathbb{AG}_N(R))$. Let $\mathcal{P}_n^* = \{A_1, A_2, \cdots, A_{2^n-2}\}$. Each G_i has exactly one element and so the Laplacian eigenvalue of each G_i is 0 and $\alpha_i = \sum_{k=1,k\neq i}^{2^n-2} d(I_{A_i}, I_{A_k}) = Tr(I_{A_i})$ for all $i = 1, 2, \cdots, 2^n - 2$. By Lemma 4.3, we have

$$M = \begin{pmatrix} Tr(I_{A_1}) & -d(I_{A_1}, I_{A_2}) & \cdots & -d(I_{A_1}, I_{A_{2^n-2}}) \\ -d(I_{A_2}, I_{A_1}) & Tr(I_{A_2}) & \cdots & -d(I_{A_2}, I_{A_{2^n-2}}) \\ \vdots & \vdots & \vdots & \\ -d(I_{A_{2^n-2}}, I_{A_1}) & -d(I_{A_{2^n-2}}, I_{A_2}) & \cdots & Tr(I_{A_{2^n-2}}) \end{pmatrix}.$$

□

Lemma 6.2. If $R = F_1 \times F_2$ then the distance Laplacian spectrum of $\mathbb{AG}_N(R)$ are $0, 2$.

Proof. We have $\mathcal{P}_2^* = \{\{1\}, \{2\}\}$, $Tr(I_{\{1\}}) = 1, Tr(I_{\{2\}}) = 1$ and $d(I_{\{1\}}, I_{\{2\}}) = 1$, By Lemma 6.1,

$$M = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

and the eigenvalues of M are $0, 2$. □

Lemma 6.3. If $R = F_1 \times F_2 \times F_3$ then the distance Laplacian spectrum of $\mathbb{AG}_N(R)$ are $11 + \sqrt{10}^{[2]}, 11 - \sqrt{10}^{[2]}, 10, 0$.

Proof. By Lemma 6.1, $M = \begin{pmatrix} 11 & -3 & -3 & -2 & -2 & -1 \\ -3 & 11 & -3 & -2 & -1 & -2 \\ -3 & -3 & 11 & -1 & -2 & -2 \\ -2 & -2 & -1 & 7 & -1 & -1 \\ -2 & -1 & -2 & -1 & 7 & -1 \\ -1 & -2 & -2 & -1 & -1 & 7 \end{pmatrix}$

and the eigenvalues of M are $11 + \sqrt{10}^{[2]}, 11 - \sqrt{10}^{[2]}, 10, 0$.

□

Lemma 6.4. If $R = F_1 \times F_2 \times F_3 \times F_4$ then the distance Laplacian spectrum of $\mathbb{AG}_N(R)$ are $3(9 + \sqrt{6})^{[3]}, 3(9 - \sqrt{6})^{[3]}, 27 + \sqrt{22}, 27 - \sqrt{22}, 28^{[2]}, 26^{[3]}, 0$.

Proof. By Lemma 6.1,

$$M = \begin{pmatrix}
31 & -3 & -3 & -3 & -2 & -2 & -2 & -3 & -3 & -3 & -2 & -2 & -2 & -1 \\
-3 & 31 & -3 & -3 & -2 & -3 & -3 & -2 & -2 & -3 & -2 & -2 & -1 & -2 \\
-3 & -3 & 31 & -3 & -3 & -2 & -3 & -2 & -3 & -2 & -2 & -1 & -2 & -2 \\
-3 & -3 & -3 & 31 & -3 & -3 & -2 & -3 & -2 & -2 & -1 & -2 & -2 & -2 \\
-2 & -2 & -3 & -3 & 25 & -2 & -2 & -2 & -2 & -1 & -2 & -2 & -1 & -1 \\
-2 & -3 & -2 & -3 & -2 & 25 & -2 & -2 & -1 & -2 & -2 & -1 & -2 & -1 \\
-2 & -3 & -3 & -2 & -2 & -2 & 25 & -1 & -2 & -2 & -1 & -2 & -2 & -1 \\
-3 & -2 & -2 & -3 & -2 & -2 & -1 & 25 & -2 & -2 & -2 & -1 & -1 & -2 \\
-3 & -2 & -3 & -2 & -2 & -1 & -2 & -2 & 25 & -2 & -1 & -2 & -1 & -2 \\
-3 & -3 & -2 & -2 & -1 & -2 & -2 & -2 & -2 & 25 & -1 & -1 & -2 & -2 \\
-2 & -2 & -2 & -1 & -2 & -2 & -1 & -2 & -1 & -1 & 19 & -1 & -1 & -1 \\
-2 & -2 & -1 & -2 & -2 & -1 & -2 & -1 & -2 & -1 & -1 & 19 & -1 & -1 \\
-2 & -1 & -2 & -2 & -1 & -2 & -2 & -1 & -1 & -2 & -1 & -1 & 19 & -1 \\
-1 & -2 & -2 & -2 & -1 & -1 & -1 & -2 & -2 & -2 & -1 & -1 & -1 & 19
\end{pmatrix}$$

and the eigenvalues of M are $3(9+\sqrt{6})^{[3]}, 3(9-\sqrt{6})^{[3]}, 27+\sqrt{22}, 27-\sqrt{22}, 28^{[2]}, 26^{[3]}, 0$.

□

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