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Research Article

Exploring Linear Algebra Through Magic Squares

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ARTICLEINFO	ABSTRACT
	India has a rich history of enigmas and mathematical challenges, as highlighted in the National Education Policy of 2020 (NEP 2020). The concept of Magic Squares (MSS) can be traced back to ancient texts such as the Garga Samhita and Nagarjuna's Kaksaputa. Notably, the final chapter of the Ganita Kaumudi, authored by Narayana Pandita in 1356 A.D., delves into the subject of MSS and their various forms. A magic square is essentially an n×n grid. It's classified as odd when n is an odd number, doubly even when n equals 4k, and singly even when n equals 4k+2, where k is a natural number. In simpler terms, a magic square can be described as a matrix of numbers. In this paper, we explore specific magic squares that have been transformed into linear equations and analyze the resulting outcomes.

INTRODUCTION

The National Education Policy of 2020 places a strong emphasis on cultivating children's interest in mathematics through games of strategy, logic, word puzzles, and recreational mathematics. It acknowledges India's rich tradition of riddles and mathematical puzzles. Fostering an enjoyable learning experience through fun exercises, games, and puzzles spanning various subjects is a vital aspect of keeping students engaged in school.

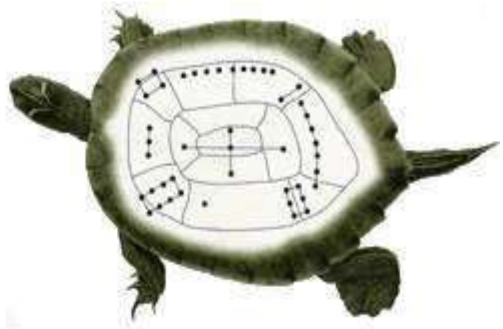
Magic squares have captivated mathematicians and even astrologers for centuries. There is substantial evidence suggesting the Indian origin and antiquity of magic squares. Ongoing research efforts explore the application of magic squares in diverse branches of mathematics and other fields.

As per Sweeney J.F. (2015), magic squares play a significant role in natural processes related to the formation of matter. Linear algebra, a branch of mathematics, deals with mathematical structures, systems of linear equations, matrices, determinants, matrix properties, linear transformations, and more.

With these considerations in mind, this paper centers on the utilization of magic squares as a means to inspire inquiries into certain linear algebra concepts. These include solving systems of linear equations and analyzing aspects of matrices, such as determinants, rank, and inverses.

For centuries, numbers have held a special significance for various cultures, often believed to possess magical properties. Specific numbers were thought to have unique qualities. For example, the number four represented the earth, given its perceived four corners, while seven was considered lucky, and thirteen unlucky.

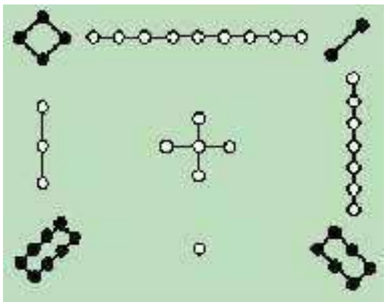
A prime example of the "magic" inherent in numbers is the concept of a magic square. Magic squares made their first recorded appearance in ancient China. Legend has it that around 2200 B.C., Emperor Yu discovered a tortoise with a peculiar pattern on its shell while strolling along the Yellow River. He named this distinctive diagram "Loh-Shu."



The Loh-Shu Tortoise (1.1)

The first recorded magic square, known as the "Scroll of the River Loh" or "Loh-Shu," was attributed to Fuh-Hi. It's a 3x3 magic square that uses symbols instead of numbers (refer to Figure 1.2). Contemporary Chinese scholars have managed to trace the Loh-Shu back to as early as the fourth century B.C., and from that point until the tenth century, it held great symbolic significance.

This early Loh-Shu square was numerical, with the number of dots in each symbol representing a whole number (refer to Figure 1.2). The even numbers were associated with the female principle, yin, while the odd numbers represented the male principle, yang. The number 5, positioned in the middle, symbolized the earth, around which the other four elements revolved: metal (represented by 4 and 9), fire (2 and 7), water (1 and 6), and wood (3 and 8).



4	9	2
3	5	7
8	1	6

The Loh-Shu Magic Square (Fig 1.2)

Magic squares have a rich history that spans various cultures and time periods. Greek writings dating back to around 1300 B.C. mention magic squares. It is believed that magic squares were introduced to Indian culture from China, where the first magic square of order four was discovered. In India, magic squares were not only used in traditional mathematical contexts but also found applications in fields such as perfume-making recipes and medical work.
Islamic and Arabic mathematicians became aware of magic squares, likely through Indian influence, by the fifth century A.D. They are often attributed to using magic squares in astrology and predictions. These mathematicians developed rules for creating larger-order magic squares and compiled a list of magic squares up to order nine. Around 1300 A.D., the Byzantine mathematician Manual Moschopoulos wrote a book based on the findings of Al-Buni, an Arab mathematician, introducing magic squares to Europe. In Europe, magic squares became associated with divination, alchemy, and astrology.

Throughout history, magic squares have been explored in relation to celestial bodies, art, and religion. They held spiritual significance in African culture, often appearing on masks, clothing, religious artifacts, and influencing house design and construction.

In India, a procedure known as the Vedic Method has been used for constructing magic squares for centuries. Hindu tradition attributes them to the deity Siva and refers to them as Bhadra Ganita. Systematic construction of magic squares based on mathematical principles was undertaken in France in the 7th century A.D. It gained popularity in Arab countries in the 10th century A.D.

India boasts several famous magic squares, including Sree Rama Chakra of the 4th order and one found in the work of Varahamihira. Various magic squares with unique properties are known, such as Eulerian Magic Squares, Kubera Chakra, Mahadeva Sooris Magic Square, Mars Square, Nasik Magic Square, Claude Gaspar Bachet Magic Square, Millennium 2000, Topsyturvy Magic Squares, Ramanujan's Square, Khajuraho Squares, Wishing Caps, Albrochet Durer's Melancholia I, and more.

Beyond their recreational and mythological aspects, magic squares exhibit advanced mathematical properties, some of which are discussed in detail.

INDIAN MAGIC SQUARES:

The compilers of ancient Hindu scriptures attributed the results and the efficacy of wearing magic squares to the Seer Garga, providing one of the earliest examples of this practice. An instance of this is found in the creation of amulets associated with Mars, intended to counteract or ward off evil effects. These amulets, crafted using specific magic squares, were believed to possess protective and beneficial properties, and their origins can be traced back to the influence of Seer Garga in Hindu tradition.

8	3	10
9	7	5
4	11	6

Neelakanta Somayaji, in his treatise "Tantra Sangraha" dated to 1444 A.D., demonstrated the Vedic basis for magic squares by drawing upon verses from the Rigveda, specifically Rigveda 10.114. He cited evidence from Seer Parasara and the Garga Samhitha to support his findings. These ancient sources served as the foundation for introducing the principles behind magic squares of order 3, as detailed in those verses.
Additionally, Acharya Nagarjuna, who lived around 100 A.D., provided rules for constructing 4x4 magic squares, accommodating both even and odd totals. This work, often attributed to Wm. Goonetilleke in 1882, involves filling eight blank squares with numbers (constants) by decoding mnemonic verses. Nagarjuna's rules and methodologies contributed significantly to the understanding and construction of magic squares in various contexts.
Using Katapayadi system

	1		8
	9		2
6		3	
4		7	

This is common for any magic constant (sum) either even or odd . The process for filling up the balance eight blank spaces for a given magic constant is also given by Nagarjuna. There is a difference in filling up them for even and odd sums.

Magic constant 2m (even)

m-3	1	m-6	8
m-7	9	m-4	2
6	m-8	3	m-1
4	m-2	7	m-9

Magic constant 2m+1 (odd)

m-3	1	m-5	8
m-6	9	m-4	2
6	m-7	3	m-1
4	m-2	7	m-8

Varahamihira(499 AD) gives a 4 x 4 magic square as follows for magic constant 18

2	3	5	8
5	8	2	3
4	1	7	6
7	6	4	1

Nasik squares , Khajuraho , Jhansi , Gwalior inscriptions are some evidences to prove the popularity of magic squares across India . But study of magic squares with theoretical mathematics background was first done by Narayana Pandita (1356 A.D)

He devoted an entire chapter for magic squares in his Ganita Kaumudi. Varieties of 4x4 magic squares , construction by super position , 4n x 4n mss, (4n+2) x (4n+2) mss , Visam (odd) squares , magic rectangles , magic hexagons, magic circles , magic triangles and some more magic figures are dealt in the book .
Narayana's methods are more elegant and practical , and centuries before western scholars who worked in this field.

Types of Magic Squares:

Certain kinds of magic squares have been given more narrow definitions based on the kinds of additional properties they possess. Listed below are some of the more common ones. Not all of these will be discussed later. There are still other kinds of magic squares besides these, but the ones listed here are among those more commonly mentioned.
A *diabolic, pan diagonal,* or *perfect magic square* is a magic square with the additional property that the sum of any extended diagonal parallel to the main diagonal and back diagonal is also the magic constant. An example will be constructed later in this paper.
A *symmetric magic square*, in addition to being magic, has the property that "the sum of the two numbers in any two cells symmetrically placed with respect to the centre cell is the same" . A symmetric magic square is also called an *associative magic square.*
A *concentric,* or *bordered,* magic square, is a magic square for which removing the top and bottom rows and the left and right columns (the "borders") results in another magic square. In the bordered square below, each of the three outer borders may be removed, leaving a square that is still magic.
A *zero magic square* is a magic square whose magic constant is 0. The set of all such zero magic squares of order n is symbolized 0MS (n). Obviously, a zero magic square cannot also be a normal magic square since it must contain negative entries. An example of Zero magic square is

Algebraic Structure

Algebraic structures consist of a set together with one or more binary operations, which are required to satisfy certain axioms.
Magic square multiplication: If the first magic square, A, is n×n, and the second one, B, is m×m. The 'product' of A and B, A*B, will be a nm× nm magic square.
Magic square addition: This operation is same as the matrix addition (component wise addition)

System of equations

For example, how we would find values for x, y and z that solve the following three equations simultaneously:

$$2x + 2y - z = -4$$
$$x + y - 4z = 3 \qquad \text{(System 1)}$$
$$5x - 3y - 3z = -2$$

The process is completed by rewriting the problem with matrices.

$$\begin{bmatrix} 2 & 2 & -1 \\ 1 & 1 & -4 \\ 5 & -3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ -2 \end{bmatrix}$$

The first matrix A is called the matrix of coefficients. The second matrix X is called the variable vector (a vector is simply a matrix having only one column). Finally, the matrix on the right-hand side can be simply called the right-hand vector, and we name it vector B. So the representation of this problem is:

$$A \cdot X = B$$

One strategy to solve is to find the inverse matrix A^{-1} (if it exists) and perform a left-multiplication on both sides (matrix multiplication is not commutative, that is, order is important, the matrix product $E \cdot F$ may differ from the product $F \cdot E$): $A^{-1} \cdot A \cdot X = A^{-1} \cdot B$. So the result is $X = A^{-1} \cdot B$.
Rank of a matrix
Definition 1
A set of vectors is linearly independent in a Vector Space if no vector in the set is:

- a scalar multiple of another vector in the set; or
- a linear combination of other vectors in the set, that is, if one vector is equal to the sum of scalar multiples of other vectors.

The rank of a matrix is the maximum number of linearly independent rows. In practice, the rank of an $n \times m$ matrix will be n if and only if all rows are linearly independent. That is, if there is no row that can be written as a linear combination of the others.
In a system of equations, each equation can be seen as a piece of information. Specifically, the left-handed side of each equation will provide a different row in the coefficient matrix, A. If a row can be written as a linear combination of the others, then in practice it means that a piece of information is being somehow repeated.
Take, for instance, the previous system and add a fourth equation
$8x - 8z = -3$:

$$\begin{aligned} 2x + 2y - z &= -4 \\ x + y - 4z &= 3 \\ 5x - 3y - 3z &= -2 \\ 8x - 8z &= -3 \end{aligned} \qquad \textit{(system 2)}$$

It looks like that the new equation adds a new piece of information. However, it does not, the fourth equation is simply the addition of all initial three equations:
Equation 4 = Equation 1 + Equation 2 + Equation 3
So technically, this fourth equation is redundant.
The following result is valid. It is mentioned in the introductory module, but its proof is only presented later in the course.

Theorem

Let A be $n \times n$ matrix, X be a vector with n variables, and B be a vector with n numbers. Then the system of equations represented by $A \cdot X = B$ will have a
unique solution if and only if rank(A) = n.

Inverse of a matrix

Our approach to solve a linear system of equations relies on finding the inverse of a matrix. Let's start with some definitions (remember that the main diagonal is the one where the row position is the same as the column position).

Definition

A $n \times n$ matrix is called the $n \times n$ *identity* if and only if all entries are zero, except on the main diagonal, which displays only ones.
This name is consistent with its main property: an identity matrix times any other matrix will be equal to the given matrix, provided the multiplication can be done.
Not all matrices have inverse, only the ones having non-zero determinant. We will avoid the formal definition of the determinant (that implies notions of permutations) for now and we will concentrate instead on its use.
Only square matrices (the same number of rows and columns) may have inverse. An equivalent way of determining the ones with inverse is stating that, from all $n \times n$ matrices, only the ones with 'rank' equal to n will have inverse.
The process of finding the inverse of a matrix can be elaborated. For our initial goals, we may use computational tools that will find them.

Theorem

Let A be a square matrix n × n whose rank is exactly equal to n, then the inverse matrix of A exists, represented by A–1. Furthermore, if X is a vector of n variables, and B is a vector with n numbers, then the system of equations represented by A · X = B has a unique solution given by $X = A^{-1} \cdot B$.

Consider the 3 × 3 magic square

From the above discussion, we want to display numbers 1, 2,... ,9 in such way that all rows, all columns and both diagonals have sum equal to 15:

a	*B*	c
d	*E*	*f*
g	*H*	*i*

The equations are:

$a + b + c = 15$ (first row)
$d + e + f = 15$ (second row)
$g + h + i = 15$ (third row)
$a + d + g = 15$ (first column)
$b + e + h = 15$ (second column)
$c + f + i = 15$ (third column)
$a + e + i = 15$ (main diagonal)
$c + e + g = 15$ second diagonal)

There are only eight equations for nine variables. We may want to create a ninth equation. We can accomplish this by giving value to a variable. For instance, let us say that e = 5, which is a reasonable number since it is the middle value and the middle cell. After adding this piece of information, we compute the rank (through an app, calculator or available websites), and we get rank = 7.
That is, from the 9 rows, we can eliminate two that are results of others. And we add two new values to variables. Without going on details of how to do, we point out that adding the first three rows gets the same result as adding rows 4, 5, and 6. So, we can eliminate one of these rows. Eliminate sixth row (basically, we noticed that row 1 + row 2 + row 3 – row 4 – row 5 = row 6).
Let us replace it by giving the value 1 to a cell. Do we want this low value on one of the corners, or in the middle? Let us try on the middle, for instance
b = 9 (that is the sixth equation now).
Again, without many details at this point, we notice that adding rows 5, 7 and 8 we end up with: {1, 1, 1, 0, 3, 0, 1, 1, 1} which is the same as adding rows 1 and 3, and three times row 9. So: row 5 + row 7 + row 8 = row 1 + row 3 + 3 × row 9.
Hence, we can eliminate one of these rows and replace it with a row that represents giving value to a new variable (since we already gave value to b and e, we can give value to any other variable other than h, since $b + e + h = 15$). Say we eliminate 8th row and let a = 4. The coefficient matrix (which we call A), the variable vector (X) and the right-handed vector (B) become:

$$A=\begin{bmatrix}1&1&1&0&0&0&0&0&0\\0&0&0&1&1&1&0&0&0\\0&0&0&0&0&0&1&1&1\\1&0&0&1&0&0&1&0&0\\0&1&0&0&1&0&0&1&0\\0&0&0&0&0&1&0&0&0\\1&0&0&0&1&0&0&0&1\\0&0&1&0&1&0&1&0&0\\0&0&0&1&0&0&0&0&0\end{bmatrix}\quad X=\begin{bmatrix}a\\b\\c\\d\\e\\f\\g\\h\\i\end{bmatrix}\quad B=\begin{bmatrix}15\\15\\15\\15\\15\\9\\15\\15\\15\end{bmatrix}$$

Now, A has rank 9. Hence, we could use linear algebra and solve it. So, our magic square is the result of $A^{-1}B$ which is:

4	9	2
3	5	7
8	1	6

How did we guarantee that the numbers did not repeat? Well, that can be done by trial-and-error or little experience. Also notice that any 90 degree rotation or any vertical/horizontal reflection or combinations of these moves would also result in similar magic squares.
Notice the following nice properties (Benjamin & Yasuda, 1999) of the above square:
- Rows: $816^2 + 357^2 + 492^2 = 618^2 + 753^2 + 294^2$
- Columns: $834^2 + 159^2 + 672^2 = 438^2 + 951^2 + 376^2$.

Conclusion

The idea for this paper was conceived upon hearing a remark that the set of magic squares forms a Group, vector space , matrix algebra, it seemed natural to consider magic squares as matrices and to investigate eigenvalues and eigenvectors and spectral radius. Once again, it is surprising that only one reference to eigenvalues of magic squares was found, and that was only a passing remark that the one of the eigenvalues of third and fourth order squares seems to be the magic constant. The preposition has turned out correct for over forty different squares of varying types of construction, but that does not constitute a proof.

Although there are a few applications of magic squares, they perhaps best belong to the category of recreational mathematics. For those who dabble in mathematics for enjoyment, magic squares are rich with mathematical properties related to many branches of mathematics. The squares were thought to be mysterious and magic, although now it is clear that they are just ways of arranging numbers and symbols using certain rules. They can be applied to Sudoku as has been discussed but are mainly of interest in mathematics for their “magic” properties.

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