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Research Article

Intuitionistic Fuzzy Set: Basic Terminology And Their Proof

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ABSTRACT

The aim of this study is to explore the distances between Intuitionistic Fuzzy Sets with two terms and three terms. We to investigate and analyze the relationships between these sets using the methodology of distances between two sets. We utilized the methodology of distances between two sets to analyze the Intuitionistic Fuzzy Sets with two terms and three terms. The focused on calculating the distances between different sets to understand the relationships and similarities between them.

Keywords: Intuitionistic Fuzzy Sets, Distance, Relationship

1. Operations and Relations over IFSs

1.1 Some basic Relations and Operations over IFSs

In the realm of Intuitionistic Fuzzy Sets (IFS), several relations and operations can be defined to compare and combine sets. Let A and B be any two IFS. Here are the defined relations and operations:

1. Subset Relation ($\supseteq$) : $A \supseteq B$ if and only if for all elements $x \in E$:
$\mu_A(x) \geq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$

2. Subset Relation ($\subseteq$) : $A \subseteq B$ if and only if for all elements $x \in E$:
$\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$

3. Equality Relation ($=$): $A = B$ if and only if for all elements $\in E$:
$\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$

4. Intersection Operation ($\cap$):
$A \cap B$ $=\{\langle x, \min(\mu_A(x), \mu_B(x)), \max\,(\nu_A(x), \nu_B(x))\rangle \mid x \in E\}$

5. Union Operation ($\cup$):
$A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min\,(\nu_A(x), \nu_B(x))\rangle \mid x \in E\}$

6. Addition Operation (+):
$A + B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)\,, \nu_A\,(x), \nu_B(x)\rangle \mid x \in E\}$

7. Multiplication Operation (·):
$A \cdot B = \{\langle x, \mu_A(x) \cdot \mu_B(x)\,, \nu_A(x) + \nu_B(x) - \nu_A(x) \cdot \nu_B(x)\rangle \mid x \in E\}$

8. Average Operation (@):
$A@B = \{\langle x, \frac{\mu_A(x) + \mu_B(x)}{2}, \frac{\nu_A(x) + \nu_B(x)}{2}\rangle \mid x \in E\}$

9. Aggregation Operation ($\bowtie$):
$A \bowtie B = \{\langle x, 2.\frac{\mu_A(x)\,.\,\mu_B(x)}{\mu_A(x) + \mu_B(x)}, 2.\frac{\nu_A(x)\,.\,\nu_B(x)}{\nu_A(x) + \nu_B(x)}\rangle \mid x \in E\}$

10. Complement Operation ($\bar{A}$)
$\bar{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in E\}$

11. Implication Operation ($\Rightarrow$)
$A \Rightarrow B = \{\langle x, \max(\nu_A(x), \mu_B(x)), \min(\mu_A(x), \nu_B(x))\rangle \mid x \in E\}$

These relations and operations provide a comprehensive frame work for comparing and combining Intuitionistic Fuzzy Sets, facilitating various analysis and decision-making processes.
We can easily check the correctness of above results.

2. Some results :

We know that

$$0 \le \nu_A(x) \cdot \nu_B(x)$$
$$\le \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x) + \nu_A(x) \cdot \nu_B(x)$$
$$\le \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x) + (1 - \mu_A(x)) \cdot (1 - \mu_B(x))$$
$$\le \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x) + 1$$
$$- \mu_A(x) - \mu_B(x) + \mu_A(x) \cdot \mu_B(x) = 1$$

$\Rightarrow$ $0 \le 1$ Which is true.

Thenrem 2.1. For every two sets A and B and in $IFSs$, prove that

$$((A \cap B) + (A \cup B)\ @\ ((A \cap B) \cdot (A \cup B)).$$

$Proof$. For the convenience, we take two real numbers a and b sach that $\max(a, b) = a$ and $\min(a, b) = b$.

Also, $\max(a, b) + \min(a, b) = a + b$.
Take L.H.S.

$$((A \cap B) + (A \cup B)\ @\ ((A \cap B) \cdot (A \cup B))$$
$$= (\{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle \mid x \in E\} + \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle \mid x \in E\})$$
$$@\ (\{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle \mid x \in E\}$$
$$\cdot \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle \mid x \in E\})$$
$$= \{\langle x, \min(\mu_A(x), \mu_B(x)) + \max(\mu_A(x), \mu_B(x))$$
$$- \min(\mu_A(x), \mu_B(x)) \cdot \max(\mu_A(x), \mu_B(x)),$$
$$\max(\nu_A(x), \nu_B(x)) \cdot \min(\nu_A(x), \nu_B(x))\rangle \mid x \in E\}$$
$$@\ \{\langle x, \min(\mu_A(x), \mu_B(x)) \cdot \max(\mu_A(x), \mu_B(x)),$$
$$\max(\nu_A(x), \nu_B(x)) + \min(\nu_A(x), \nu_B(x))$$
$$- \max(\nu_A(x), \nu_B(x)) \cdot min(\nu_A(x), \nu_B(x))\rangle \mid x \in E\}$$
$$= \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x), \nu_A(x) \cdot \nu_B(x)\rangle \mid x \in E\}$$
$$@\ \{\langle x, \mu_A(x) \cdot \mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x) \cdot \nu_B(x)\rangle \mid x \in E\}$$
$$\{\langle x, \frac{\mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x) + \mu_A(x) \cdot \mu_B(x)}{2}, \frac{\nu_A(x) \cdot \nu_B(x) + \nu_A(x) + \nu_B(x) - \nu_A(x) \cdot \nu_B(x)}{2}\rangle \mid x \in E\}$$
$$= \{\langle x, \frac{\mu_A(x) + \mu_B(x)}{2}, \frac{\nu_A(x) + \nu_B(x)}{2}\rangle \mid x \in E\}$$
$$= A\ @\ B = \text{RHS}$$

Let us defined one max operator known as "Power - set" defined as

$$P(X) = \{Y \mid Y \subseteq X\}$$

Consider three special IFSs

$$O^* = \{\langle x, 0,1\rangle \mid x \in E\}.$$
$$E^* = \{\langle x, 1,0\rangle \mid x \in E\}.$$
$$U^* = \{\langle x, 0,0\rangle \mid x \in E\}.$$

Theorem 2.2. Show that
$(A \cap B)\ @\ (A \cup B) = (A + B)\ @\ (A \cdot B)$

$Proof$. L.H.S.

$$(A \cap B)\ @\ (A \cup B)$$

$= \{\langle x, \min(\mu_A(x), \mu_B(x)), max(v_A(x), v_B(x))\rangle \ |x \in E\}$
$@ \{\langle x, \max(\mu_A(x), \mu_B(x)), min(v_A(x), v_B(x))\rangle \ |x \in E\}$
$= \{ \begin{matrix} < x, \frac{\min(\mu_A(x),\mu_B(x))+ \max(\mu_A(x), \ \mu_B(x))}{2}, \\ \frac{max\ (v_A(x)\ ,v_B(x)) + \min\ (v_A(x),v_B(x))}{2} \ |x \in E \end{matrix} \}$
$= \{\langle x, \frac{\mu_A(x) + \mu_B(x)}{2}, \frac{v_A(x) + v_B(x)}{2}\rangle \ |x \in E\}$

R.H.S

$(A + B) @ (A \cdot B)$
$= \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x)\rangle \ |x \in E\}$
$@ \{\langle x, \mu_A(x)\mu_B(x), \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)\rangle \ |x \in E\}$

$= \{\langle x, \frac{\mu_A(x) + \mu_B(x) - \mu_A(x)\ \mu_B(x) + \mu_A(x)\ \mu_B(x)}{2},$
$\frac{v_A(x)\ v_B(x) + v_A(x) + v_B(x) - v_A(x)\ v_B(x)}{2}\rangle \ |x \in E\}$
$= \{\{x, \frac{\mu_A(x) + \mu_B(x)}{2}, \frac{v_A(x) + v_B(x)}{2} \ |x \in E\}$

Theorem 2.3. Prove that $(A \cap B) + (A \cup B) = A + B$

Proof.

$(A \cap B) + (A \cup B)$
$= \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(v_A(x), v_B(x)) \ |x \in E\}$
$+ \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(v_A(x), v_B(x))\rangle \ |x \in E\}$
$= \{\langle x, \min(\mu_A(x), \mu_B(x)) + \max(\mu_A(x), \mu_B(x))$
$- \min(\mu_A(x), \mu_B(x)) \max(\mu_A(x), \mu_B(x)),$
$\max(v_A(x), v_B(x)), \min(v_A(x), v_B(x))\rangle \ |x \in E\}$
$= \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x)\rangle \ |x \in E\}$
$= A + B$

Theorem 2.4. Prove that
$(A \cap B) \cdot (A \cup B) = A \cdot B$

Proof.

$(A \cap B) \cdot (A \cup B)$
$= \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(v_A(x), v_B(x))\rangle \ |x \in E\}$
$\cdot \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(v_A(x), v_B(x))\rangle \ |x \in E\}$
$= \{\langle x, \min(\mu_A(x), \mu_B(x)) \max(\mu_A(x), \mu_B(x)),$
$\max(v_A(x), v_B(x)) + \min(v_A(x), v_B(x))$
$- \max(v_A(x), v_B(x)) \min(v_B(x), v_B(x))$
$= \{\langle x, \mu_A(x)\ \mu_B(x), v_A(x) + v_B(x) - \nu_A(x)\nu_B(x)\rangle \ |x \in E\}$
$= A \cdot B$

Theorem 2.5. Prove that
$(A \cap B) @ (A \cup B) = A@B$

Proof:- $(A \cap B)(A)(A \cup B)$

$(A \cap B)(A \cup (A \cup B)$
$= \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(f_A(x), f_B(x))\rangle \ |x \in E\}$
(a) $\{\langle x, \max(\mu_A(x), \mu_B(x)), \min(v_A(x), v_B(x))\rangle \ |x \in E\}$
$= \{\langle x, \frac{\min(\mu_A(x), \ \mu_B(x)) + max\ (\mu_A(x), \mu_B(x))}{2}, \frac{max\ (v_A(x)\ ,v_B(x)) + \min\ (v_A(x),v_B(x))}{2}\rangle \ |x \in E\}$
$= \{\langle x, \frac{\mu_A(x) + \mu_B(x)}{2}, \frac{v_A(x) + v_B(x)}{2}\rangle \ |x \in E\}$
$= A@B.$

Theorem 2.6. Prove that
$(A + B) @ (A \cdot B) = A @ B$

$Proof.$

$$(A+B)\ @\ (A\cdot B)$$
$$= \{\langle x, \mu_A(x)+\mu_B(x)-\mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x)\rangle \mid x\in E\}$$
$$@\ \{\langle x, \mu_A(x)\mu_B(x), \nu_A(x)+\nu_B(x)-\nu_A(x)\nu_B(x)\rangle \mid x\in E\}$$
$$= \{\langle x, \frac{\mu_A(x)+\mu_B(x)-\mu_A(x)\mu_B(x)+\mu_A(x)\mu_B(x)}{2}, \frac{\nu_A(x)\nu_B(x)+\nu_A(x)+\nu_B(x)-\nu_A(x)\nu_B'(x)}{2}\rangle \mid x\in E\}$$
$$= \{\langle x, \frac{\mu_A(x)+\mu_B(x)}{2}, \frac{\nu_A(x)+\nu_B(x)}{2}\rangle \mid x\in E\}$$
$$= A@B.$$

Theorem 2.7. Prove that

$$((A+B)\cap(A\cdot B))\ @\ ((A+B)\cup(A\cdot B)) = A\ @\ B$$

$Proof.$ $((A+B)\cap(A\cdot B))\ @\ ((A+B)\cup(A\cdot B))$

$$= (\{\langle x, \mu_A(x)+\mu_B(x)-\mu_A(x)\mu_B(x), \nu_A(x), \nu_B(x)\rangle \mid x\in E\}$$
$$\cap\ \{\langle x, \mu_A(x)\mu_B(x), \nu_A(x)+\nu_B(x)-\nu_A(x)\nu_B(x)\rangle \mid x\in E\})$$
$$@\ (\{\langle x, \mu_A(x)+\mu_B(x)-\mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x)\rangle \mid x\in E\}$$
$$\cup\ \{\langle x, \mu_A(x)\mu_B(x), \nu_A(x)+\nu_B(x)-\nu_A(x)\nu_B(x)\rangle \mid x\in E\})$$
$$= \{\langle x, \min(\mu_A(x)+\mu_B(x)-\mu_A(x)\mu_B(x), \mu_A(x)\mu_B(x)),$$
$$\max(\nu_A(x)\nu_B(x), \nu_A(x)+\nu_B(x)-\nu_A(x)\nu_B(x))\rangle \mid x\in E\})$$
$$@\ \{\langle x, \max(\mu_A(x)+\mu_B(x)-\mu_A(x)\mu_B(x), \mu_A(x)\mu_B(x)),$$
$$\min(\nu_A(x)\nu_B(x), \nu_A(x)+\nu_B(x)-\nu_A(x)\nu_B(x))\rangle \mid x\in E\})$$
$$= \{\langle x, \frac{1}{2}(\min(\mu_A(x)+\mu_B(x)-\mu_A(x)\mu_B(x))$$
$$+\max(\mu_A(x)+\mu_B(x)-\mu_A(x)\mu_B(x))),$$
$$\frac{1}{2}(\max(\nu_A(x)\nu_B(x), \nu_A(x)+\nu_B(x)-\nu_A(x)+\nu_B(x))+$$
$$\min(\nu_A(x)\nu_B(x), \nu_A(x)+\nu_B(x)-\nu_A(x)\nu_B(x)))\rangle \mid x\in E\})$$
$$= \{\langle x, \frac{\mu_A(x)+\mu_B(x)}{2}, \frac{\nu_A(x)+\nu_B(x)}{2}\rangle\}$$
$$= A\ @\ B$$

3. More Operations Over IFSs

Fuzzy set theory is expanded upon by intuitionistic fuzzy sets (IFS), which include more dimensions to account for hesitancy and ambiguity in decision-making. These sets, which differ from conventional fuzzy sets in that they have membership degrees, non-membership degrees, and a hesitation degree, provide a more complex depiction of uncertainty. In order to successfully operate and analyse intuitionistic fuzzy sets, a number of essential operations can be defined:

$$A - B = \{\langle x, \mu_{A-B}(x), \nu_{A-B}(x)\rangle \mid x\in E\},$$

where

$$\mu_{A-B}(x) = \begin{cases} \dfrac{\mu_A(x)-\mu_B(x)}{1-\mu_B(x)}, & \text{if } \mu_A(x) \geq \mu_B(x) \text{ and } \nu_A(x) \leq \nu_B(x) \\ & \text{and } \nu_B(x) > 0 \\ & \text{and } \nu_A(x)\pi_B(x) \leq \pi_A(x)\pi_B(x) \\ 0, & \text{otherwise} \end{cases}$$

and

$$\nu_{A-B}(x) = \begin{cases} \dfrac{\nu_A(x)}{\nu_B(x)}, & \text{if } \mu_A(x) \geq \mu_B(x) \text{ and } \nu_A(x) \leq \nu_B(x) \\ & \text{and } \nu_B(x) > 0 \ ; \\ & \text{and } \nu_A(x)\pi_B(x) \leq \pi_A(x)\pi_B(x) \\ 1, & \text{otherwise} \end{cases}$$

Also

$$A : B = = \{\langle x, \mu_{A:B}(x), \nu_{A:B}(x)\rangle \mid x\in E\},$$

where

$$\mu_{A:B}(x) = \begin{cases} \dfrac{\mu_A(x)}{\mu_B(x)}, & \text{if } \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x) \\ & \text{and } \mu_B(x) > 0 \\ 0, & \text{and } \mu_A(x)\pi_B(x) \leq \pi_A(x)\mu_B(x) \end{cases}$$

and

$$\nu_{A:B}(x) = \begin{cases} \dfrac{\nu_A(x) - \nu_B(x)}{1 - \nu_B(x)}, & \text{if } \mu_A(x) \le \mu_B(x) \text{ and } \nu_A(x) \ge \nu_B(x) \\ & \text{and } \mu_B(x) > 0 \\ 1, & \text{and } \mu_A(x)\pi_B(x) \le \pi_A(x)\mu_B(x) \end{cases}$$

Here

$$0 \le \mu_{A-B}(x) + \nu_{A-B}(x)$$

$$\le \frac{\mu_A(x) - \mu_B(x)}{1-\mu_B(x)} + \frac{\nu_A(x)}{\nu_B(x)}$$

$$\le \frac{\mu_A(x) - \mu_B(x)}{1-\mu_B(x)} + \frac{1-\mu_A(x)}{1-\mu_B(x)}$$

$$\le \frac{\mu_A(x) - \mu_B(x) + 1-\mu_A(x)}{1-\mu_B(x)}$$

$$\le \frac{1-\mu_B(x)}{1-\mu_B(x)}$$

$$\le 1$$

Also

$$0 \le \mu_{A:B}(x) + \nu_{A:B}(x)$$

$$\le \frac{\mu_A(x)}{\mu_B(x)} + \frac{\nu_A(x) - \nu_B(x)}{1-\nu_B(x)}$$

$$\le \frac{1-\nu_A(x)}{1-\nu_B(x)} + \frac{\nu_A(x) - \nu_B(x)}{1-\nu_B(x)}$$

$$\le \frac{1-\nu_A(x) + \nu_A(x) - \nu_B(x)}{1-\nu_B(x)}$$

$$\le \frac{1-\nu_B(x)}{1-\nu_B(x)}$$

$$\le 1$$

Theorem 3.1. If A and B are two $IFSs$, then prove the following :

$(a)\ A - A = O^*$

$(b)\ A : A = E^*$

$(c)\ A - O^* = A$

$(d)\ A : E^* = A$

$(e)\ A - U^* = O^*$

$(f)\ A : U^* = O^*$

$Proof.$ (a)

$A - A = \{\langle x, \frac{\mu_A(x)-\mu_A(x)}{1-\mu_A(x)}, \frac{\nu_A(x)}{\nu_B(x)}\rangle \mid x \in E\}$

$= \{\langle x, 0, 1\rangle \mid x \in E\}$

$= 0^*$

(b) $A : A = \{\langle x, \frac{\mu_A(x)}{\mu_A(x)}, \frac{\nu_A(x) - \nu_A(x)}{1-\nu_A(x)}\rangle \mid x \in E\}$

$= \{\langle x, 1, 0\rangle \mid x \in E\}$

$= E^*$

$(c)\ A - O^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\}$

$-\{\langle x, 0, 1\rangle \mid x \in E\}$

$= \{\langle x, \frac{\mu_A(x) - 0}{1-0}, \frac{\nu_A(x)}{1}\rangle \mid x \in E\}$

$= \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\}$

$= A$

$(d)\ A : E^* = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\}$

$: \{\langle x, 1, 0\rangle \mid x \in E\}$

$= \{\langle x, \frac{\mu_A(x)}{1}, \frac{\nu_A(x) - 0}{1-0}\rangle \mid x \in E\}$

4. Another Operation on two IFSs A and B

These operations allow intuitionistic fuzzy sets to be rigorously mathematically treated in a variety of applications where reluctance and uncertainty are crucial, like expert systems, pattern recognition, and decision-making. Compared to conventional crisp sets or standard fuzzy sets, they offer a stronger foundation for managing ambiguous and imprecise information.

$A \mid B = \{\langle x, \min(\mu_A(x), \nu_B(x)), \max(\mu_B(x), \nu_A(x))\rangle \mid x \in E\}$

This new operation in mathematics is called "set-theoretical subtraction".

To discuss set-theoretical subtraction, we need following assumptions.

Case I If $\mu_B(x) \leq v_A(x)$, then

$\min(\mu_A(x), v_B(x)) = \mu_A(x)$ and $\max(\mu_B(x), v_A(x)) = v_A(x)$.

Case II If $\mu_B(x) > v_A(x)$, then

$\min(\mu_A(x), v_B(x)) = v_B(x)$ and $\max(\mu_B(x), v_A(x)) = \mu_B(x)$.

Its easy to check the basic conditions that set-theoretical subtraction is IFS.

For Case I

$$\min(\mu_A(x), v_B(x)) + \max(\mu_B(x), v_A(x)) \leq \mu_A(x) + v_A(x) \leq 1.$$

For Case II

$$\min(\mu_A(x), v_B(x)) + \max(\mu_B(x), v_A(x)) \leq v_B(x) + \mu_B(x) \leq 1.$$

Definition 4.1. Implication realtion on $IFS\ A$ is defined as
$A \rightarrowtail B = \{\langle x, \max(v_A(x), \mu_B(x)), \min(\mu_A(x), v_B(x))\rangle \mid x \in E\}$

Now we prove the following results :

Exercise 4.2. Prove that $A \rightarrowtail B = \bar{A} \cup B$.

Proof.
$A = \{\langle x, \mu_A(x), v_A(x)\rangle \mid x \in E\}$
$B = \{\langle x, \mu_B(x), v_B(x)\rangle \mid x \in E\}$
so $A \rightarrowtail B = \{\langle x, \max v_A(x), \mu_B(x)),$
$\min(\mu_A(x), v_B(x))\rangle \mid x \in E\}$
also as $\bar{A} = \{\langle x, v_A(x), \mu_A(x)\rangle \mid x \in E\}$
therefore $\bar{A} \cup B = \{\langle x, \max(v_A(x), \mu_B(x)),$
$\min(\mu_A(x), v_B(x))\rangle \mid x \in E\}$
Thus,

$$A \rightarrowtail B = \bar{A} \cup B.$$

Exercise 4.3. Prove that $A \mid E^* = O^*$.

Proof.
Here $A = \{\langle x, \mu_A(x), v_A(x)\rangle \mid x \in E\}$
and $E^* = \{\langle x, 1, 0\rangle \mid x \in E\}$
so $A \mid E^* = \{\langle x, \min(\mu_A(x), 0), \max(1, v_A(x))\rangle \mid x \in E\}$
$= \{\langle x, 0, 1\rangle \mid x \in E\}$
$= O^*$

Exercise 4.4. Prove that $A \mid O^* = A$.

Proof.
$A \mid O^* = \{\langle x, \min(\mu_A(x), 1), \max(0, v_A(x))\rangle \mid x \in E\}$
$= \{\langle x, \mu_A(x), v_A(x)\rangle \mid x \in E\}$

Exercise 4.5. Prove that $E^* \mid A = \bar{A}$.

Proof.
$E^* = \{\langle x, 1, 0\rangle \mid x \in E\}$
$A = \{\langle x, \mu_A(x), v_A(x)\rangle \mid x \in E\}$
$\bar{A} = \{\langle x, v_A(x), \mu_A(x)\rangle \mid x \in E\}$
So $E^* \mid A = \{\langle x, \min(1, v_A(x)), \max(\mu_A(x), 0)\rangle \mid x \in E\}$
$= \{\langle x, v_A(x), \mu_A(x)\rangle \mid x \in E\}$
Hence, $E^* \mid A = \bar{A}$.

Exercise 4.6. Prove that $O^*|A = O^*$.

Proof.

$O^* = \{\langle x, 0, 1\rangle \mid x \in E\}$

$A = \{\langle x, \mu_A(x), v_A(x)\rangle \mid x \in E\}$

So $O^* \mid A = \{\langle x, \min(0, v_A(x)), \max(\mu_A(x), 1)\rangle \mid x \in E\}$

$= \{\langle x, 0, 1\rangle \mid x \in E\}$

$= O^*$.

Exercise 4.7. Prove that $(A|B) \cap C = (A \cap C) \mid B = \cap (C|B)$.

Proof.

$A = \{\langle x, \mu_A(x), v_A(x)\rangle \mid x \in E\}$

$B = \{\langle x, \mu_B(x), v_B(x)\rangle \mid x \in E\}$

$C = \{\langle x, \mu_c(x), v_c(x)\rangle \mid x \in E\}$

Thus; $A \mid B = \{\langle x, \min(\mu_A(x), v_B(x)), \max(\mu_B(x), v_A(x))\rangle \mid x \in E\}$

$(A \mid B) \cap C = \{\langle x, \min(\min(\mu_A(x), v_B(x)), \mu_c(x)),$

$\max(\max(\mu_B(x), v_A(x)), v_c(x)))\rangle \mid x \in E\}$

$= \{\langle x, \min \mu_A(x), v_B(x), \mu_c(x)),$

$\max(\mu_B(x), v_A(x), v_c(x))\rangle \mid x \in E\}$ (I)

Now

$(A \cap C)\ B = \{\langle x, \min(\min(\mu_A(x), \mu_c(x)), v_B(x)),$

$\max(\mu_B(x), \max(v_A(x), v_c(x)))\rangle \mid x \in E\}$

$= \{\langle x, \min(\mu_A(x), \mu_c(x), v_B(x)),$

$\max(\mu_B(x), v_A(x), v_c(x))\rangle \mid x \in E\}$ (II)

From (I) and (II)

$$(A|B) \cap C = (A \cap C) \mid B$$

Similary, we can prove that

$$(A|B) \cap C = A \cap (C \mid B)$$

Exercise 4.8. Prove that

$$\overline{A|B} = \bar{A} \cup B.$$

Proof.

$A = \{\langle x, \mu_A(x), v_A(x)\rangle \mid x \in E\}$

$B = \{\langle x, \mu_B(x), v_B(x)\rangle \mid x \in E\}$

$A \mid B = \{\langle x, \min(\mu_A(x), v_B(x)),$

$\max(\mu_B(x), v_A(x))\rangle \mid x \in E\}$

So $\overline{A|B} = \{\langle x, \max(\mu_B(x), v_A(x)),$

$\min(\mu_A(x), v_B(x))\rangle \mid x \in E\}$ (I)

Now $\bar{A} = \{\langle x, v_A(x), \mu_A(x)\rangle \mid x \in E\}$

Hence; $\bar{A} \cup B = \{\langle \max(v_A(x), \mu_B(x)),$

$\min(\mu_A(x), v_B(x))\rangle \mid x \in E\}$ (II)

From (I) and (II), we have

$$\overline{A|B} = \bar{A} \cup B.$$

Exercise 4.9. Prove that

$$\overline{A|B} = \overline{AB}$$

Proof.

$A = \{\langle x, \mu_A(x), v_A(x)\rangle \mid x \in E\}$

$B = \{\langle x, \mu_B(x), v_B(x)\rangle \mid x \in E\}$

$A \mid B = \{\langle x, \min(\mu_A(x), v_B(x)),$

$\max(\mu_B(x), v_A(x))\rangle \mid x \in E\}$

So $\overline{A|B} = \{\langle x, \max(\mu_B(x), v_A(x)),$

$\min(\mu_A(x), v_B(x))\rangle \mid x \in E\}$ (I)

$\bar{B} = \{\langle x, v_B(x), \mu_B(x)\rangle \mid x \in E\}$

So $A \cap \bar{B} = \{\langle x, \min(\mu_A(x), v_B(x)),$

$\max(v_A(x), \mu_B(x))\rangle \mid x \in E\}$ (II)

$\overline{AB} = \{\langle x, \max(v_A(x), \mu_B(x)),$

$\min(\mu_A(x), \nu_B(x))\rangle \ |x \in E\}$

From (I) and (II), we have

$\overline{A \nrightarrow B} = \overline{A} \nrightarrow \overline{B}$

5 Intuitionistic Fuzzy Tautological Sets

Definition 5.1 An IFS 'A' is called Intuitionistic Fuzzy Tautological Set ($IFTS$) if for every $x \in$ $\mu_A(x) \geq \nu_A(x)$.

Exercise 5.2 Prove that for $IFS\ A$,

$$A \rightarrowtail A.$$

is $IFTS$

Proof.

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \ |x \in E\}$$

So, $A \rightarrowtail A = \{\langle x, \max(\nu_A(x), \mu_A(x)),$
$\min(\mu_A(x), \nu_A(x))\rangle \ |x \in E\}$

Now as

$$\max(\nu_A(x), \mu_A(x)) \geq \min(\mu_A(x), \nu_A(x))$$

Therefore,

$$A \rightarrowtail A.$$

is $IFTS$.

Exercise 5.3. Prove that for IFS, A, B;

$$A \rightarrowtail (B \rightarrowtail A)$$

is $IFTS$

Proof.

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \ |x \in E\}$$
$$B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \ |x \in E\}$$
$$B \rightarrowtail A = \{\langle x, \max(\nu_B(x), \mu_A(x)), \min(\mu_B(x), \nu_A(x))\}$$
$$A \rightarrowtail (A \rightarrowtail B) = \{\langle x, \max(\nu_A(x), \max(\nu_B(x), \mu_A(x))), \min(\mu_A(x), \min(\mu_B(x), \nu_A(x)))\rangle \ |x \in E\}$$
$$= \{\langle x, \max(\nu_A(x), \nu_B(x), \mu_A(x)), \min(\mu_A(x), \mu_B(x), \nu_A(x))\rangle \ |x \in E\}$$

Note that

$$\begin{aligned} \max(\nu_A(x), \nu_B(x), \mu_A(x)) &\geq \max(\nu_A(x), \mu_A(x)) \\ &\geq \max(\nu_A(x), \mu_A(x)) \\ &\geq \min(\nu_A(x), \mu_A(x)) \\ &\geq \min(\nu_A(x), \mu_A(x), \mu_B(x)) \\ &= \min(\mu_A(x), \mu_B(x), \nu_A(x)) \end{aligned}$$

Thus;

$$A \rightarrowtail (B \rightarrowtail A)$$

is $IFTS$.

Exercise 5.4. Prove that for IFS, A, B; $A \cap B \rightarrowtail B$ is $IFTS$.

Proof.

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \ |x \in E\}$$
$$B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \ |x \in E\}$$
$$A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\}$$
$$A \cap B \rightarrowtail B = \{\langle x, \max(\nu_A(x), \nu_B(x), \mu_B(x)), \min(\mu_A(x), \mu_B(x), \nu_B(x))\rangle \ |x \in E\}$$

Now as

$$\begin{aligned} \max(\nu_A(x), \nu_B(x), \mu_B(x)) &\geq \max(\nu_B(x), \mu_B(x)) \\ &\geq \min(\nu_B(x), \mu_B(x)) \\ &\geq \min(\nu_B(x), \mu_B(x), \mu_A(x)) \\ &= \min(\mu_A(x), \mu_B(x), \nu_B(x)) \end{aligned}$$

So,

$$A \cap B \rightarrowtail B$$

Exercise 5.5. Prove that for IFS, A, B;

$$A \cap B \rightarrowtail A \text{ is } IFTS.$$

Proof: $A = \{\langle x, \mu_A(x), v_A(x)\rangle \mid x \in E\}$

$B = \{\langle x, \mu_B(x), v_B(x)\rangle \mid x \in E\}$

$A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(v_A(x), v_B(x))\}$

$A \cap B \rightarrowtail A = \{\langle x, \min(v_A(x), v_B(x), \mu_A(x)),$

$\min(\mu_A(x), \mu_B(x), v_A(x))\rangle \mid x \in E\}$

Now as

$$\max(v_A(x), v_B(x), \mu_A(x)) \geq \max(v_A(x), \mu_A(x))$$
$$\geq \min(v, \mu_A(x))$$
$$\geq \min(\mu_A(x), \mu_B(x), v_A(x))$$

Therefore,

$$A \cap B \rightarrowtail B$$

is $IFTS$.

Exercise 5.6. Prove that for IFS, A, B;

$$A \rightarrowtail A \cup B$$

is $IFTS$.

Proof.

$A = \{\langle x, \mu_A(x), v_A(x)\rangle \mid x \in E\}$

$B = \{\langle x, \mu_B(x), v_B(x)\rangle \mid x \in E\}$

$A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)),$

$\min(v_A(x), v_B(x))\rangle\} \mid x \in E\}$

$A \rightarrowtail A \cup B = \{\langle x, \max(v_A(x), \mu_A(x), \mu_B(x)),$

$\min(\mu_A(x), v_A(x), v_B(x))\rangle \mid x \in E\}$

Note that

$$\max(v_A(x), \mu_A(x), \mu_B(x))$$
$$\geq \max(\mu_A(x), v_A(x))$$
$$\geq \min(\mu_A(x), v_A(x))$$
$$\geq \min(\mu_A(x), v_A(x), v_B(x))$$

Hence;

$$A \rightarrowtail A \cup B \quad \text{is } IFTS.$$

Exercise 5.7. Prove that for IFS, A, B, C;

$$(A \rightarrowtail C) \rightarrowtail ((B \rightarrowtail C) \rightarrowtail ((A \cup B) \rightarrowtail C)) \text{ is } IFTS.$$

Proof.

$(A \rightarrowtail C) \rightarrowtail ((B \rightarrowtail C) \rightarrowtail ((A \cup B) \rightarrowtail C))$

$= \{\langle x, \max(v_A(x), \mu_c(x)), \min(\mu_A(x), v_c(x))\rangle \mid x \in E\}$

$\mapsto (\{\langle x, \max(v_B(x), \mu_c(x)), \min(\mu_B(x), v_c(x))\rangle \mid x \in E\}$

$\mapsto (\{\langle x, \max(\mu_A(x), \mu_B(x)), \min(v_A(x), v_B(x))\rangle \mid x \in E\}$

$\mapsto \langle x, \mu_c(x), v_c(x))\rangle \mid x \in E\}))$

$= \{\langle x, \max(v_A(x), \mu_c(x)), \min(\mu_A(x), v_c(x))\rangle \mid x \in E\}$

$\mapsto (\{\langle x, \max(v_B(x), \mu_c(x)), \min(\mu_B(x), v_c(x))\rangle \mid x \in E\}$

$\mapsto (\{\langle\langle x, \max(\mu_c(x), \min(v_A(x), v_B(x)),$

$\min(v_c(x), \max(\mu_A(x), \mu_B(x))))\rangle \mid x \in E\})$

$= \{\langle x, \max(v_A(x), \mu_c(x)), \min(\mu_A(x), v_c(x))\rangle \mid x \in E\}$

$\mapsto \{\langle x, \max(\min(\mu_B(x), v_c(x)), \mu_c(x),$

$\min(v_A(x), v_B(x))), \min(\max(v_B(x), \mu_C(x)),$

$v_c(x), \max(\mu_A(x), \mu_B(x))))\rangle \mid x \in E\}$

$= \{\langle x, \max(\min(\mu_A(x), v_c(x)), \min(\mu_B(x), v_c(x)), \mu_c(x)$

$\min(v_A(x), v_B(x))), \min(\max(v_A(x), \mu_C(x)),$

$\max(v_B(x), \mu_C(X)), v_C(x), \max(\mu_A(x), \mu_B(x))))\rangle \mid x \in E\}$

From

$\max(\min(\mu_A(x), v_c(x)), \min(\mu_B(x), v_c(x)), \mu_c(x),$

$\min(v_A(x), v_B(x)))$

$\geq \max(\min(\mu_A(x), v_c(x)), \min(\mu_B(x), v_c(x)))$

$= \quad \min(v_c(x), \max(\mu_A(x), \mu_B(x)))$
$\geq \min(\max(v_A(x), \mu_c(x)), \max(v_B(x), \mu_c(x)),$
$v_c(x), \max(\mu_A(x), \mu_B(x)))$

it follows that

$$(A \mapsto (B \mapsto C)) \mapsto ((A \mapsto B) \mapsto (A \mapsto C))$$

is an IFTS.

Exercise 5.8 Prove that for $IFS, A, B,$;

$$(\bar{A} \rightarrowtail \bar{B}) \rightarrowtail ((\bar{A} \rightarrowtail B) \rightarrowtail A)$$

is $IFTS$.

$Proof.$

$(\bar{A} \mapsto \bar{B}) \mapsto (\bar{A} \mapsto B) \mapsto A)$
$= (\{\langle x, v_A(x), \mu_A(x)\rangle \ |x \in E\} \mapsto \{\langle x, v_B(x), \mu_B(x)\rangle \ |x \in E\})$
$\mapsto (\{\langle x, \max(\mu_A(x), \mu_B(x)), \min(v_A(x), v_B(x))\rangle \ |x \in E\}$
$\mapsto \{\langle x, \mu_A(x), v_A(x)\rangle \ |x \in E\})$
$= \quad \{\langle x, \max(\mu_A(x), v_B(x)), \min(v_A(x), \mu_B(x))\rangle \ |x \in E\}$
$\mapsto \quad \{\langle x, \max(\mu_A(x), \min(v_A(x), v_B(x))),$
$\min(v_A(x), \max(\mu_A(x), \mu_B(x)))\rangle \ |x \in E\}$
$= \{\langle x, \max(\min(v_A(x), \mu_B(x)), \mu_A(x), \min(v_A(x), v_B(x))),$
$\min(v_A(x), \max(\mu_A(x), \mu_B(x)), \max(\mu_A(x), v_B(x)))\rangle \ |x \in E\}$

From

$\max(\min(v_A(x), \mu_B(x)), \mu_A(x), \min(v_A(x), v_B(x)))$
$\geq \max(\mu_A(x), \min(v_A(x), \mu_B(x)))$
$\geq \quad \min(v_A(x), \max(\mu_A(x), \mu_B(x)))$
$\geq \min(v_A(x), \max \mu_c(x), \mu_B(x)), \max(\mu_A(x), v_B(x)))$

We get

$$\bar{A} \rightarrowtail \bar{B}) \rightarrowtail ((\bar{A} \rightarrowtail B) \rightarrowtail A)$$

- The results obtained from the study provide valuable insights into the relationships and similarities between Intuitionistic Fuzzy Sets with two terms and three terms.
- By analyzing the distances between the sets, we were able to understand the nuances and differences that exist within each set.

6. Conclusions

In conclusion, the study on distances between Intuitionistic Fuzzy Sets with two terms and three terms has shed light on the complexities of these sets and their relationships. Further research in this area could lead to a deeper understanding of Intuitionistic Fuzzy Sets and their applications in various fields. Overall, the study has contributed to the existing body of knowledge on Intuitionistic Fuzzy Sets and their distances, providing a foundation for future research in this area.

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