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Higher-Order Generalized Invexity And Strict Minimizers In Vector Optimization Within Conic Spaces

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ARTICLE INFO	**ABSTRACT**

Optimisation theory is a cornerstone of engineering and technology, providing essential tools and methodologies for enhancing performance, efficiency, and functionality across various applications. This paper delves into the vector optimisation problem within the context of cones, a critical area that underpins many advanced engineering and technological processes. To this aim, Higher order Kstrict minimizers and higher order strongly K-non smooth invex functions and its generalizations are defined for a vector optimization problem over cones and Kuhn Tucker type necessary optimality condition are established for a K-strict minimizers of higher order. Further, these K-strict minimizers of higher order are characterized via sufficient optimality conditions by utilizing the above functions. Finally a Mond-Weir type dual is formulated and corresponding duality results are obtained.

Keywords: K-strict minimizers of order r, strong K-non smooth invexity of order r,K-generalized invexity
MSC Classification: 90C29 , 90C46 , 90C25 , 90C26

1 Introduction

In the present time, Vector optimization problem over cones has great importance in the field of optimization theory. The concepts of efficiency, weak efficiency and proper efficiency are related to the solution of vector optimization problem have introduced in the literature [7, 10]. The concept of generalizing local minimizers is essential for understanding the convergence of iterative numerical methods and their stability outcomes [5, 8]. Bhatia [1] elaborates on Ward's concept by introducing the idea of an enhanced global stringent minimizer for multiobjective optimization problems. Furthermore, Sahay and Bhatia [2, 6] have broadened the definition of rigorous minimizers, applying it to both scalar and multiobjective optimization scenarios.
Nowadays research has made significant strides in this area. For instance, the invex functions have found wider acceptance among the researchers working in optimization theory. In recent years, a lot of literature [16, 17, 19, 20, 22, 23, 25–33] has appeared on invexity and generalized invexity notion. Nanda et al. [26] established duality results for a pair of non-linear programming problems under the concept of generalized ρ-(η,θ)-invexity. The notion of (G-V, ρ)-invexity [23] was introduced to derive sufficiency and optimality conditions. By utilizing the concept of (Φ,ρ)-invexity, Upadhyay et al. [33] studied a non-differentiable minimax fractional programming problem. Singh et al. [28] considered a multiobjective variational problem and discussed duality results by introducing the idea of second order (Φ,ρ)-invexity for continuous case. Optimality and duality results were established under non-differentiable vectorial $(\Phi, \rho)^w$-invexity notion by Antczak et al. [17]. Under the notion of higher order generalized invexity, Al-Homidian et al. [14] studied a non-differentiable minimax fractional programming problem. By introducing the concept of $B-(H_p,r,\alpha)$-invexity, Liu and Yuan [23] derived optimality results for a mathematical programming problem. Stancu Minasian et al. [30] introduced two new classes of generalized invex functions namely, (p, r) $-\rho-(\eta,\theta)$-quasi-invex and (strictly) $(p,r)-\rho-(\eta,\theta)$-pseudo-invex functions and employed these to prove duality theorems for a minimax fractional programming problem. Das and Nahak [18] introduced the notion of $(p,r)-\rho-(\eta,\theta)$-invexity for set valued optimization problems and derived optimality and duality results. Under

the notion of B-type I functions, Kumar et al. [22] presented optimality and duality results for multiobjective semi-infinite variational problem. The concept of invexity of order $\sigma(B,\phi)$-V-type II was introduced by An and Gao [17] as a generalization of invexity.

The authors utilized newly introduced functions to establish optimality and duality results for a nonsmooth multiobjective programming problem. In [29], StancuMinasian et al. introduced the notion of higher-order (ϕ,ρ)-V-invexity and studied a semi-infinite minimax fractional programming problem. Moreover, they discussed various duality results under the assumption of higher-order (ϕ,ρ)-V-invexity. Further, Rusu-Stancu and Stancu-Minasian [30] introduced higher-order (ϕ,ρ)-V-type I invex functions and used them to establish duality results for Wolfe higher-order type multiobjective dual programs. Gut´ıerrez et al. [20] conducted a study on vector critical points and efficiency in vector optimisation with Lipschitz functions, utilising pseudoinvexity hypotheses. Kanzi [21] studied a non-differentiable multiobjective semi-infinite optimization problem and proved sufficient optimality condition under invexity assumptions. Suneja et al. [31] formulated the condition for optimal solution in nonsmooth vector optimization problems having generalized cone-invex objective and constraint functions.

Liu et al. [24] proposed multiple descriptions to approximate the solution sets for nonsmooth optimisation problems using generalised strongly preinvex functions.

The present study is important due to more than one reason. First, our study generalizes the work of several previous works. Second, not many researchers have utilized strict global minimizers over cone as a solution concept in the recent past. Therefore, the paper will bridge the gap in this direction. So, the paper can act as a reference material as well. The results of this paper will enrich the existing literature on one hand, while on the other, it will also help and motivate researchers working in the area of generalized invexity. Theoretical foundations of vector optimisation, as shown in [12], have been further developed by research into higher-order efficiency criteria. Additionally, the existence and uniqueness of solutions have been explored through studies on generalised invex functions, as discussed in [13]. Moreover, the integration of conic space theory with vector optimization has opened new avenues for solving complex multi-objective problems described in [14].

In the realm of environmental engineering, these concepts could be applied to optimize various processes and systems aimed at addressing environmental challenges. For instance, consider the design of a wastewater treatment plant where multiple objectives need to be optimized, such as minimizing energy consumption, maximizing pollutant removal efficiency, and reducing operational costs.

This research work presents a novel notion called a higher order K-strict minimizer for a vector optimisation problem over cones, which takes a further step in this approach. We introduce specific generalisations of higher-order strong non-smooth invexity over cones in order to analyse this solution notion. The solution notion is characterised by these generalisations, which provide sufficient optimality criteria. A dual of the Mond-Weir type is formulated and well-known results of duality are produced.

2 Preliminaries

Let $S \subseteq R^n$ be an open subset of R^n.

Definition 1. *A real valued function $\phi: S \to R$ is said to be locally Lipschitz at $u \in S$ if there exists a positive constant ℓ such that*

$$|\phi(x) - \phi(\bar{x})| \leq \ell \|x - \bar{x}\|, \text{for all } x,\bar{x} \in N_\delta(u),$$

where $N_\delta(u)$ is some neighborhood of u.

Definition 2. *The Clarke's generalized directional derivative of $\phi: S \to R$ at $x \in S$ in the direction $v \in R^n$, denoted by $\phi^\circ(x;v)$ is defined as follows:*

$$\phi^\circ(x;v) = \lim_{\substack{y \to x \\ t \to 0^+}} \sup \frac{\phi(y + tv) - \phi(y)}{t},$$

where y is a vector in S and t is a positive scalar.

Definition 3. *Let $\phi: S \to R$ be locally Lipschitz, then the Clarke's generalized gradient of ϕ at $x \in S$, denoted by $\partial\phi(x)$ is given by*

$$\partial\phi(x) = \{\xi \in R^n \mid \phi^\circ(x;v) \geq \langle \xi,v \rangle, \text{for all } v \in R^n\}.$$

Let $f : S \to R^m$ be a vector valued function given by $f = (f_1,f_2,...f_m)$, where each f_i is a real valued function defined on S. Then f is said to be locally Lipschitz on S if each f_i is locally Lipschitz on S. The generalized directional derivative of a locally Lipschitz function $f : S \to R^m$ at $x \in S$ in the direction $v \in R^n$ is given by

$$f^\circ(x,v) = \{f_1^\circ(x;v), f_2^\circ(x;v), \ldots, f_m^\circ(x;v)\}.$$

The generalized gradient of f at x is the set

$\partial f(x) = \partial f_1(x) \times \cdots \times \partial f_m(x)$,

where $\partial f_i(x)$ is the generalized gradient of f_i at x for $i = 1,2,...m$.

Every element $A = (v_1,...,v_m) \in \partial f(x)$ is a continuous linear operator from S to R^m and

$Ax = (\langle v_1,x \rangle,...\langle v_m,x \rangle) \in R^m$, for all $x \in R^n$.

The following result has been proved by Clarke [3].

Theorem 1. *1. If $f_i : R^n \to R$ is locally Lipschitz function then for each $x \in R^n$*

$f_i^\circ(x;v) = \max\{\langle \xi, v\rangle \mid \xi \in \partial f_i(x)\}$, *for every* $v \in R^n$, $i = 1,2,\ldots, m$.

2. If f_i $(i = 1,2,\ldots,m)$ *is a finite family of functions on* R^n *each of which is locally Lipschitz then* $\sum_{i=1}^{m} f_i$ *is also locally Lipschitz and*

$$\partial\left(\sum_{i=1}^{m} f_i\right)(x) \subseteq \sum_{i=1}^{m} \partial f_i(x), \quad \text{for every } x \in R^n.$$

We now give the definitions of K-generalized invex and K-nonsmooth invex functions defined by Yen and Sach [11] and the relations between them.

Let $S \subseteq R^n$ be an open set. Let $f : S \to R^m$ be a locally Lipschitz function on R^n and let $K \subseteq R^m$ be a closed convex cone with nonempty interior. Then $K^+ = \{y^* \in R^m : \langle y,y^*\rangle \geq 0$, for all $y \in K\}$.

Definition 4 ([11])**.** *f is said to be K-generalized invex at the point* $u \in S$ *if there exists* $\eta : S \times S \to R^n$ *such that for every* $x \in S$ *and* $A \in \partial f(u)$,

$f(x) - f(u) - A\eta(x,u) \in K$.

Definition 5 ([11])**.** *f is said to be K-nonsmooth invex at* $u \in S$, *if there exists* $\eta : S \times S \to R^n$ *such that for every* $x \in S$,

$f(x) - f(u) - f^\circ(u;\eta) \in K$,

where $f^\circ(u;\eta) = \{f_1^\circ(u;\eta), f_2^\circ(u;\eta), \ldots, f_m^\circ(u;\eta)\}$.

Remark 1 ([9])**.** *If f is K-generalized invex at u with respect to* $\eta : S \times S \to R^n$ *then f is K-nonsmooth invex at u with respect to same* η. *But the converse is not true.* In this paper, we study the following vector optimization problem over cones:

(VP) K-minimize $f(x)$ subject to $-g(x) \in Q$

$-h(x) \in P$

where $f : S \to R^m$, $g : S \to R^p$, $h : S \to R^k$, $S \subseteq R^n$ is an open set. $P = \{0\}$, K and Q are closed convex pointed cones with nonempty interiors in R^k, R^m and R^p, respectively. Let $X = \{x \in S : -g(x) \in Q, -h(x) \in P\}$ denote the set of feasible solutions of (VP).

Let us introduce a new notion of strict minimizer of order r for (VP).

Definition 6. *Let* $r \geq 1$ *be an integer. A point* $x^* \in X$ *is said to be K-strict minimizer of order r for (VP) with respect to a vector valued function* $\psi : S \times S \to R^n$ *if there exists a constant* $c \in \text{int } R^m_+$ *such that*

$$f(x) - f(x^*) - c\|\psi(x,x^*)\|^r \in -/\text{int } K, \text{ for all } x \in X.$$

Let us now introduce the strongly K-nonsmooth invex function of higher order and its generalizations.

Definition 7. *Let* $S \subseteq R^n$ *be an open set. Let* $\eta : S \times S \to R^n$ *and* $\psi : S \times S \to R^n$. *Let* $f : S \to R^m$ *be a locally Lipschitz function on S. Let* $r \geq 1$ *be an integer. Then f is said to be*

1. strongly K-nonsmooth invex function of order *r with respect to* η,ψ *at* $u \in S$ *if there exists a constant* $c \in \text{int } R^m_+$ *such that for all* $x \in S$,
$$f(x) - f(u) - f^\circ(u;\eta(x,u)) - c\|\psi(x,u)\|^r \in K.$$
2. strongly K-nonsmooth pseudoinvex type I of order *r with respect to* η,ψ *at* $u \in S$ *if there exists a constant* $c \in \text{int } R^m_+$ *such that for all* $x \in S$ $f^\circ(u;\eta(x,u)) \in K$ *implies* $f(x) - f(u) - c\|\psi(x,u)\|^r \in K$,

or equivalently
$$f(x) - f(u) - c\|\psi(x,u)\|^r \in -\text{int } K \text{ implies } f^\circ(u;\eta(x,u)) \in -\text{int } K.$$
3. strongly K-nonsmooth pseudoinvex type II of order *r with respect to* η, ψ *at* $u \in S$ *if there exists a constant* $c \in \text{int } R^m_+$ *such that for all* $x \in S$, $f^\circ(u;\eta(x,u)) + c\|\psi(x,u)\|^r \in K$ *implies* $f(x) - f(u) \in K$.
4. strongly K-nonsmooth quasiinvex type I of order *r with respect to* η, ψ *at* $u \in S$ *if there exists a constant* $c \in \text{int } R^m_+$ *such that for all* $x \in S$, $f(x) - f(u) \in -K$ *implies* $f^\circ(u;\eta(x,u)) + c\|\psi(x,u)\|^r \in -K$.
5. *strongly K-nonsmooth quasiinvex type II of order r with respect to* η, ψ *at* $u \in S$ *if there exists a constant* $c \in \text{int } R^m_+$ *such that for all* $x \in S$, $f(x) - f(u) - c\|\psi(x,u)\|^r \in -K$ *implies* $f^\circ(u;\eta(x,u)) \in -K$.

Remark 2. *1. If* $K = R^+$, *then definition of K-strict minimizers of order r reduces to that of strict minimizers of order r given by Bhatia and Sahay [2].*

2. If $K = R^+$ *and f is a differentiable function on S then the definitions of strongly K-nonsmooth invex function of order r, strongly K-nonsmooth pseudoinvex type I and type II of order r and strongly K-nonsmooth quasiiinvex type I and type II of order r reduces to those of strongly invex of order r, strongly pesudoinvex type I and type II of order r and strongly quasiinvex type I and type II of order r given by Bhatia and Sahay [2].*

Remark 3. *Every strongly K-nonsmooth invex function of order r is K-nonsmooth invex function with respect to the same* η *but the converse is not true.*

Example 1. *Let* $f : R \to R^2$ *be defined by*

$$f(x) = \begin{cases} (1-2x, 0), & x < 0 \\ (1,0), & x \geq 0 \end{cases},$$

Let $\eta : R \times R \to R$ *be defined by* $\eta(x,u) = (x - u)^3$. $K = \{(x,y) : y \geq 0, x \leq y\}$. *Let* $\psi : R \times R \to R$ *be defined by* $\psi(x,u) = (4 + xu)^{\frac{1}{3}},\ r = 3$ *and* $c = (1,0)$. *At* $u = 0$

$$f^{\circ}(0;\eta(x,0)) = \begin{cases} (-2x^3, 0), & x \leq 0 \\ (0,0), & x > 0\,, \end{cases}$$

Then f is strongly K-nonsmooth invex function of order 3 at $u = 0$. *But f is not K-nonsmooth invex function at* $u = 0$ *because at* $x = -\frac{1}{10}$

$$f(x) - f(0) - f^{\circ}(0;\eta(x,0)) = \left(\frac{99}{500}, 0\right) \notin K.$$

Remark 4. *Every strongly K-nonsmooth invex function of order r is strongly Knonsmooth pseudoinvex type I of order r with respect to the same η, ψ but the converse is not true.*

Example 2. *Let* $f(x) = \begin{cases} (2+x, 4), & x < 0 \\ (2,4), & x \geq 0 \end{cases},\ \eta(x,u) = x - u,\ K = \{(x,y) : y \leq -x\}$,
$c = (0,1),\ \psi = (u+x)^{\frac{1}{2}},\ r = 2.$

At $u = 0$

$$f^{\circ}(0;\eta(x,0)) = \begin{cases} (x,0)\,, & x \leq 0 \\ (0,0)\,, & x > 0 \end{cases}$$

Then f is strongly K-nonsmooth pseudoinvex type I of order 2 at $u = 0$. *But f is not strongly K-nonsmooth invex function of order 2 at* $u = 0$ *with respect the same η and ψ because for* $x = -\frac{1}{2}$

$$f(x) - (0) - f^{\circ}(0;\eta(x,0)) - c\|\psi(x,0)\|^2 = \left(\frac{1}{2}, 0\right) \notin K\,.$$

Remark 5. *Every strongly K-nonsmooth quasiinvex type I of order r is strongly Knonsmooth invex of order r with respect to the some η, ψ but the converse is not true.* √

Example 3. *In Example 2 if we take* $c = (0,1)$ *and* $\psi = 1 + xu$. *Then f is strongly K-nonsmooth invex at* $u = 0$ *of order 2 but f is not strongly K-nonsmooth quasiinvex I at* $u = 0$ *of order 2 with respect to the same η, ψ because*
$f(x) - f(0) \in -K \Rightarrow x \leq 0$.
But for $x = -\frac{1}{2}$

$$f^{\circ}(0;\eta(x,0)) + c\|\psi(x,0)\|^r = \left(-\frac{1}{2}, 1\right) \notin -K.$$

Remark 6. *The relations between these classes of functions and some related classes are summarized in Figure 1 (Note: It is important to observe that there is no relation between type II functions and corresponding notions of type I functions presented in Figure 1).*

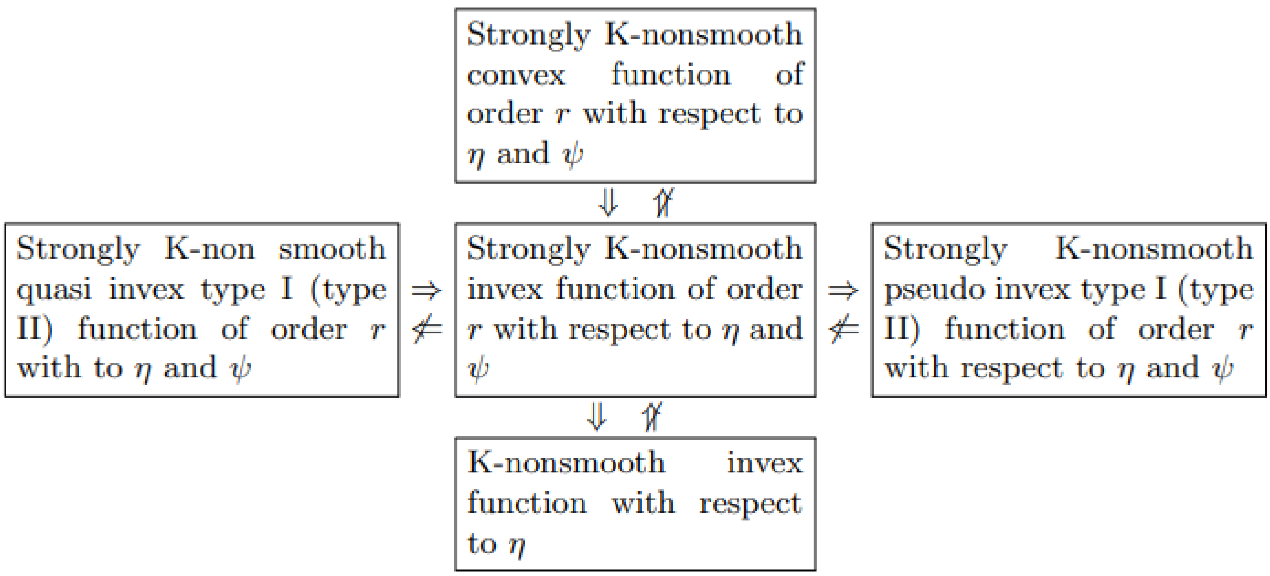

Fig. 1

3 Optimality Conditions

For each $\alpha \in K^+$, $\beta \in Q^+$ and $\gamma \in P^+$, we assume that $\alpha f = \alpha \circ f$, $\beta g = \beta \circ g$ and $\gamma h = r \circ h$ are locally Lipschitz functions.

We shall be using the following Slater constraint qualification in obtaining the necessary optimality conditions.

Definition 8. *The problem (VP) is said to satisfy the generalized Slater constraint qualification at $x^* \in X$ if there exists $\bar{x} \in S$ such that $-g(\bar{x}) \in \text{int } Q$, $-h(\bar{x}) = 0$.*

Theorem 2 (Karush-Kuhn-Tucker type necessary optimality conditions)**.** *Suppose x^* is a K-strict minimizer of order r with respect to ψ for (VP). Let f be K-generalized invex, g be Q-generalized invex and h be P-generalized invex at x^* with respect to the same η, ψ. Assume that (V P) satisfies the Slater constraint qualification at x^*. Then there exists $0 \neq \alpha \in K^+$, $\beta \in Q^+$ and $\gamma \in P^+$ such that*

$$0 \in \partial(\alpha f)(x^*) + \partial(\beta g)(x^*) + \partial(\gamma h)(x^*), \qquad \text{for all } x \in S \tag{1}$$

$$\beta g(x^*) = 0 \tag{2}$$

Proof. First, we claim that the system

$-\phi(x) \in \text{int}(K \times Q \times P)$

has no solution $x \in S$, where $\phi(x) = (f(x) - f(x^*), g(x), h(x) - h(x^*))$.

Since f is K-generalized invex, g is Q-generalized invex and h is P-generalized invex with respect to the same η it follows from the generalized alternative theorem given by Craven and Yang [4] that there exists $0 \neq \alpha \in K^+$, $\beta \in Q^+$ and $\gamma \in P^+$ not all zero such that $\alpha(f(x) - f(x^*)) + \beta g(x) + \gamma(h(x) - h(x^*)) \geq 0$, for all $x \in S$.

That is,

$$\alpha f(x) + \beta g(x) + \gamma h(x) \geq \alpha f(x^*) + \gamma h(x^*), \text{ for all } x \in S. \tag{3}$$

Taking $x = x^*$ in (3), we get

$\beta g(x^*) \geq 0$.

Since $\beta \in Q^+$ and $-g(x^*) \in Q$, we get $\beta g(x^*) \leq 0$. It follows that $\beta g(x^*) = 0$.

Thus from (3) it follows that

$(\alpha f + \beta g + \gamma h)(x) - (\alpha f + \beta g + \gamma h)(x^*) \geq 0$, for all $x \in S$, which implies that x^* is a minimum point of the problem

$\min_{x \in S}(\alpha f + \beta g + \gamma h)(x)$,

which gives that

$0 \in \partial(\alpha f + \beta g + \gamma h)(x^*)$

that is

$0 \in \partial(\alpha f)(x^*) + \partial(\beta g)(x^*) + \partial(\gamma h)(x^*)$.

Let if possible $\alpha = 0$ then from (3), we get

$$\beta g(x) + \gamma h(x) \geq \gamma h(x^*), \text{ for all } x \in S \tag{4}$$

By generalized Slater constraint qualification there exists $\bar{x} \in S$ such that $-g(\bar{x}) \in \text{int } Q, -h(\bar{x}) = 0$, which gives that

$\beta g(\bar{x}) < 0$

which contradicts (4) as $h(x^*) = h(\bar{x}) = 0$. Hence $\alpha \neq 0$. □

Now we give sufficient optimality conditions in the form of the following theorems.

Theorem 3. *Let f be strongly K-nonsmooth pseudoinvex type I of order r, g be strongly Q-nonsmooth quasiinvex type I of order r and h be strongly P-nonsmooth quasiinvex type I of order r at $x^* \in X$ with respect to the same η and ψ. If there exists $0 \neq \alpha \in K^+$, $\beta \in Q^+$ and $\gamma \in P^+$ such that* (1) *and* (2) *hold then x^* is a K-strict minimizer of order r with respect to the same ψ for (VP).*

Proof. Since (1) holds therefore there exists $x^\circ \in \partial(\alpha f)(x^*)$, $y^\circ \in \partial(\beta g)(x^*)$ and $z^\circ \in \partial(\gamma h)(x^*)$ such that

$$x^\circ + y^\circ + z^\circ = 0. \tag{5}$$

Let if possible x^* be not a K-strict minimizer of order r with respect to ψ for (VP).

Then for $c \in \text{int } R^m_+$ there exists some $x \in X$ such that

$f(x) - f(x^*) - c\|\psi(x,x^*)\|^r \in -\text{int } K$.

As f is strongly K-nonsmooth pseudoinvex type I at x^* of order r with respect to η, ψ, therefore it follows that

$f^\circ(x^*;\eta(x,x^*)) \in -\text{int } K$.

$\Rightarrow \quad \alpha f^\circ(x^*;\eta(x,x^*)) < 0$ (as $0 \neq \alpha \in K^+$)

which is equivalent to

$\alpha A\eta(x,x^*) < 0$, for all $A \in \partial f(x^*)$,

as

$$f_i^\circ(x^*;\eta(x,x^*)) = \sup\{\langle v_i, \eta(x,x^*)\rangle : v_i \in \partial f_i(x^*)\},$$

that is,

$$x^\circ\eta(x,x^*) < 0, \text{ where } x^\circ \in \partial(\alpha f)(x^*) \text{ and } \alpha > 0. \tag{6}$$

Now $x \in X$, therefore $\beta g(x) \leq 0$. From condition (2), it follows that

$\beta(g(x) - g(x^*)) \leq 0$,

which gives that

$$g(x) - g(x^*) \in -Q \qquad (\text{as } \beta \in Q^+).$$

Since g is strongly Q-nonsmooth quasiinvex type I at x^* of order r with respect to the same η, ψ so it follows that there exists a constant $c_1 \in \text{int } R^p_+$ such that

$g^\circ(x^*;\eta(x,x^*)) + c_1\|\psi(x,x^*)\|^r \in -Q$.

This implies
$\beta g^{\circ}(x^*;\eta(x,x^*)) + \beta c_1\|\psi(x,x^*)\|^r \leq 0$ (as $\beta \in Q^+$)
or
$\beta g^{\circ}(x^*;\eta(x,x^*)) \leq -\beta c_1\|\psi(x,x^*)\|^r \leq 0$.
So $\beta g^{\circ}(x^*;\eta(x,x^*)) \leq 0$,
which is equivalent to
$\beta B\eta(x,x^*) \leq 0$, for all $B \in \partial g(x^*)$,
that is

$$y^{\circ}\eta(x,x^*) \leq 0, \text{ as } \beta \geq 0 \text{ and } y^{\circ} \in \partial(\beta g)(x^*). \quad (7)$$

Again $h(x) = h(x^*)$ as $x,x^* \in X$. So $h(x) - h(x^*) \in -P$.
As h is strongly P-nonsmooth quasiinvex type I at $x^*$of order r with respect to the same η,ψ so it follows that there exists a constant $c_2 \in \text{int } R^k_+$ such that

$$h^{\circ}(x^*,\eta(x,x^*)) + c_2\|\psi(x,x^*)\|^r \in -P$$

$$\Rightarrow \quad \gamma h^{\circ}(x^*,\eta(x,x^*)) + \gamma c_2\|\psi(x,x^*)\|^r \leq 0 \qquad (\text{as } \gamma \in P^+)$$

$$\Rightarrow \quad \gamma h^{\circ}(x^*;\eta(x,x^*)) \leq 0,$$

which is equivalent to
$\gamma\, C\eta(x,x^*) \leq 0$, for all $C \in \partial h(x^*)$.
That is

$$z^{\circ}\eta(x,x^*) \leq 0, \text{ as } \gamma \geq 0 \text{ and } z^{\circ} \in \partial(\gamma h)(x^*). \quad (8)$$

Adding (7) and (8), we get
$(y^{\circ} + z^{\circ})\, \eta(x,x^*) \leq 0$,
which implies that

$$x^{\circ}\eta(x,x^*) \geq 0 \qquad (\text{as } x^{\circ} + y^{\circ} + z^{\circ} = 0),$$

which contradicts (6). Hence $x^*$is a K-strict minimizer of order r with respect to the same ψ for (VP). □

Theorem 4. *Let f be strongly K-nonsmooth quasiinvex type II of order r,g be strongly Q-nonsmooth quasiinvex type I of order r and h be strongly P-nonsmooth quasiinvex type I of order r at $x^* \in X$ with respect to the same η, ψ. If there exists $0 \neq \alpha \in K^+$, $\beta \in Q^+$ and $\gamma \in P^+$ such that* (1) *and* (2) *hold then $x^*$is a strict minimizer of order r with respect to the same ψ for (VP).*

Remark 7. *The result of Theorems 3 and 4 holds if g is strongly Q-nonsmooth invex function of order r and h is strongly P-nonsmooth invex function of order r with respect to the same η, ψ on S.*

4 Duality

We associate the following dual program with (VP).

(VD) K-maximize $f(u)$

$$\text{subject to } 0 \in \partial(\lambda f)(u) + \partial(\tau g)(u) + \partial(\mu h)(u) \quad (9)$$

$$(\tau g)(u) \geq 0 \quad (10)$$

$$(\mu h)(u) = 0 \quad (11)$$

$$0 \neq \lambda \in K^+,\ \tau \in Q^+,\ \mu \in P^+$$

Theorem 5 (Weak Duality)**.** *Let x and (u,λ,τ,μ) be feasible solutions of (VP) and (VD), respectively. Suppose f is strongly K-nonsmooth pseudoinvex type I of order r,g is strongly Q-nonsmooth quasiinvex type I of order r and h is strongly P-nonsmooth qusiinvex type I of order r at u with respect to the same η and ψ then there exists $c \in \text{int } R^m_+$ such that $f(x) - f(u) - c\|\psi(x,u)\|^r \notin$ int K.*

Proof. Since (u,λ,τ,μ) is feasible for (VD), therefore by (9) there exists $x^* \in \partial(\lambda f)(u)$, $y^* \in \partial(\tau g)(u)$ and $z^* \in \partial(\mu h)(u)$ such that

$$x^* + y^* + z^* = 0. \quad (12)$$

Let if possible, for every $c \in \text{int } R^m_+$ such that
$f(x) - f(u) - c\|\psi(x,u)\|^r \in -\text{int } K$.
Since f is strongly K-nonsmooth pseudoinvex type I of order r at u with respect to η and ψ, we get
$f^{\circ}(u;\eta(x,u)) \in -\text{int } K$.
Now $0 \neq \lambda \in K^+$, gives that
$\lambda f^{\circ}(u;\eta(x,u)) < 0$,
which is equivalent to $\lambda A\eta(x,u) < 0$, for all $A \in \partial f(u)$.
This gives that
$x^*\eta(x,u) < 0$, where $x^* \in \partial(\lambda f)(u)$ and $\lambda > 0$.
Now using (12), we get

$$-(y^* + z^*)\eta(x,u) < 0. \quad (13)$$

Since $y^* \in \partial(\tau g)(u)$. Therefore

(14)

$y^* = \tau B^*$ for some $B^* \in \partial g(u)$.
Again since $z^* \in \partial(\mu h)(u)$. Therefore

(15)

$z^* = \mu C^*$ for some $C^* \in \partial h(u)$.
From (13), (14) and (15), we have

(16)

$-(\tau B^* + \mu C^*)\eta(x,u) < 0$.
Now we claim that

(17)

$\tau g^\circ(u;\eta(x,u)) + \mu h^\circ(u;\eta(x,u)) \leq 0$.
As x is feasible for (VP) and (u,λ,τ,μ) is feasible for (VD), therefore

$$\tau g(x) \leq 0 \leq \tau g(u) \tag{18}$$

and

$$\mu h(x) = 0 = \mu h(u). \tag{19}$$

If $\tau = 0$ and $\mu = 0$ then the above inequality (17) holds trivially.
Case 1. if $\tau = 0$ and $\mu = 0$ then (17) reduces to

$$\tau g^\circ(u;\eta(x,u) \leq 0. \tag{20}$$

From (18), we get that
$g(x) - g(u) \in -Q$(as $\tau \in Q^+$)
Since g is strongly Q-nonsmooth quasiinvex type I at u of order r with respect to same η,ψ, therefore there exists a constant $c \in \text{int } R^p_+$ such that
$g^\circ(u;\eta(x,u)) + c\|\psi(x,u)\|^r \in -K$,
which implies that $\tau g^\circ(u;\eta(x,u)) - \tau c\|\psi(x,u)\|^r \geq 0$.
Thus gives
$-\tau g^\circ(u;\eta(x,u)) \geq \tau c\|\psi(x,u)\|^r \geq 0$.
That is
$-\tau g^\circ(u;\eta(x,u)) \geq 0$. This shows that (20) holds and is equivalent to
$-\tau B\eta(x,u) \geq 0$, for all $B \in \partial g(u)$,
which contradicts (16).
Hence in this case the result holds.
Case 2. Suppose $\tau = 0$ and $\mu \neq 0$
On the same lines of Case 1 we can prove that
$-\mu C\eta(x,u) \geq 0$, for all $C \in \partial h(u)$
which contradicts (16). Hence the result holds in this case also.
Case 3. Suppose $\tau \neq 0$ and $\mu \neq 0$ Since $\tau \neq 0$, then from (18)
$g(x) - g(u) \in -Q$
As done in case (1) we get

$$-\tau B\eta(x,u) \geq 0, \text{ for all } B \in \partial g(u). \tag{21}$$

Again since $\mu \neq 0$ so as done in case (2), we get

$$-\mu c\eta(x,u) \geq 0, \text{ for all } C \in \partial h(u) \tag{22}$$

Adding (21) and (22), we get that (17) holds and is equivalent to
$-(\tau B + \mu C)\eta(x,u) \geq 0$,
which contradicts (16). Hence the result holds in this case also. □

Theorem 6 (Strong Duality). *Suppose x^* is a K-strict minimizer of order r with respect to ψ for (VP). Assume that (VP) satisfies the generalized Slater constraint qualification at x^*. If f is K-generalized invex, g is Q-generalized invex and h is Pgeneralized invex at x^* of order r with respect to the same η and ψ. Then there exists $0 \neq \lambda^* \in K^+$, $\tau^* \in Q^+$ and $\mu^* \in P^+$ such that $(x^*,\lambda^*,\tau^*,\mu^*)$ is a feasible solution for (VD). Further, if the conditions of Weak Duality Theorem 5 hold for all feasible x for (VP) and all feasible (u,λ,τ,μ) for (VD) then $(x^*,\lambda^*,\tau^*,\mu^*)$ is a K-strict maximizer of order r for (VD).*

5 Discussion

This research work extends the notion of invexity to higher orders and generalizing strong non-smooth invexity over cones and offers a framework for modeling complex systems where the optimization landscape may exhibit non-smooth behavior. Furthermore, the derivation of sufficient optimality conditions tailored to this solution concept enables engineers to formulate more precise optimization algorithms and decision-making strategies. These conditions can guide the design and operation of complex systems, ensuring that they achieve desired performance levels while minimizing resource consumption.
Additionally, the formulation of a Mond-Weir type dual and the derivation of duality results provide a theoretical underpinning for assessing the trade-offs inherent in some kind of engineering decisions. This duality framework allows engineers to understand the relationships between different objectives and constraints, facilitating the identification of Pareto-optimal solutions that represent the best possible compromises among competing goals.

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