Educational Administration: Theory and Practice
2024,30(8), 593 – 600
ISSN:2148-2403
https://kuey.net/
Research Article

Simultaneous Approximation To A Interpolatory Polynomials And Its Derivative On The Roots Of Laguerre Polynomials By Pál-Type Interpolation

Vaishali Agarwal[1*],Rekha Srivastava[2]

[1*,2]Department of Mathematics and Astronomy, University of Lucknow, 226007 India. Email:[1]-vaishali3295@gmail.com; [2]-srivastava_rekha@lkouniv.ac.in

Citation:Vaishali Agarwal, et.al(2024), Simultaneous Approximation To A Interpolatory Polynomials And Its Derivative On The Roots Of Laguerre Polynomials By Pal-Type´ Interpolation, *Educational Administration: Theory and Practice, 30(8), 593 – 600*
Doi: 10.53555/kuey.v30i8.7716

ARTICLEINFO

ABSTRACT

The objective of this paper is to construct a interpolatory polynomial with Laguerre conditions based on the zeros of the polynomials $L_n^{(k)}(x)$ and $L_n^{(k)'}(x)$ where $L_n^{(k)}(x)$ is the Laguerre polynomial of degree n and the derivative of Laguerre polynomial $L_n^{(k)'}(x)$ is of degree $n-1$. A modified Pál-type interpolation problem is studied in a unified way. We prove the regularity of the problem and give the explicit formulae of the interpolation. Also, the existence and uniqueness of the polynomial is proved if the inner nodal points are the roots of the interpolatory polynomials and obtain an estimate over the whole real number line.
Mathematics Subject Classification 2000; 41A10, 97N50.

Keywords: Lagrange interpolation, Laguerre abscissas, Pál-type interpolation, Explicit form, Order of convergence.

1. Introduction

In 1975, L. G. Pál [4] has introduced a modification of the Hermite-Fejér interpolation, in which the function values and the first derivatives are prescribed on two inter scaled systems of nodal points $\{x_i\}_{i=1}^{n}$ and $\{x_i^*\}_{i=1}^{n-1}$, that is $-\infty < x_1 < x_1^* < x_2 < \cdots < x_{n-1} < x_{n-1}^* < x_n < +\infty$, where

$$w_n(x) = (x-x_1)(x-x_2)\ldots(x-x_n) \text{ and } w_n'(x) = n(x-x_1^*)\ldots(x-x_{n-1}^*).$$

He proved that for any given systems of real numbers $\{u_k\}_{k=1}^{n}$ and $\{u_k'\}_{k=1}^{n-1}$, there exists a polynomial $Q_{2n-1}(x)$ of minimal degree $(2n-1)$ satisfying the following interpolational properties:

$$Q_{2n-1}(x_k) = u_k \qquad (k = 1, 2, \ldots, n)$$
$$Q_{2n-1}'(x_k^*) = u_k' \qquad (k = 1, 2, \ldots n-1).$$

This interpolational polynomial is not uniquely determined; hence for the uniqueness an additional condition is recommended. Introducing the additional condition $Q_{2n-1}(x_0) = 0$ at an additional knot $x_0 \neq x_k \ \ (k = 1, 2, \ldots, n)$, Pál proved the uniqueness and gave an explicit formula for it. Following Pál's idea many authors researched this kind of interpolation and they called it Pál-type interpolation. In 2004, Lénárd [2] investigated the Pál-type interpolation problem on the nodes of Laguerre abscissas. Pál demonstrated that there is no distinct polynomial of degree $\leq 2n-2$ when function values are dictated on one set of n points and derivatives values on another set of $n-1$ points, but note that there is a unique polynomial having the degree $\leq 2n-1$when function value is defined at one more point that does not belong to the previous collection of n points.

Srivastava [6] studied special problem of mixed type weighted interpolation on the mixed zeros of Hermite polynomial and its derivative. She proved the existence, uniqueness and convergence of the theorem. Srivastava studied an interpolation on the polynomials with Hermite conditions on the zeros of Ultra spherical polynomial at the closed interval −1 to 1. They have proved the existence, uniqueness, explicit representation and convergence theorem of the interpolatory polynomials, which are the zeros of the ultra spherical polynomial of degree *n*. Srivastava and Singh [7] have proved existence, uniqueness, explicit representation and order of convergence of the interpolatory problem when the roots are given on the ultra spherical polynomial with boundary conditions on the closed interval −1 to 1.

Many authors [1], [3], [5], [8], and [10] have studied about interpolation problems when the function values and its sequential derivatives are specified at the provided set of points. When using lacunary interpolation, non-consecutive derivatives are used in the interpolation procedure to retrieve the data. The function values in Pál-type interpolation are specified at the zeros of $w_n(x)$, whereas the derivative values are specified at the zeros of $w_n'(x)$.

In this paper we study the following interpolation problem: On the infinite interval $[0,\infty)$ let $\{x_i\}_{i=0}^{n}$ and $\{y_i\}_{i=1}^{n}$ be the arbitrary two sets of inter scaled nodal points:

$$0 \le x_0 < y_1 < x_1 < \ldots < x_{n-1} < y_n < x_n < +\infty. \tag{1}$$

For any fixed integer $k \ge 1$, obtain a least degree polynomial $P_m(x)$ satisfying the $(0;1)$ interpolation conditions

$$P_m(x_i) = z_i, \quad (i = 1, \ldots, n) \tag{2}$$

$$P_m'(y_i) = z_i' \quad (i = 1, \ldots, n-1) \tag{3}$$

with Hermite-type boundary conditions

$$P_m^{(j)}(x_0) = z_0^{(j)}, \quad (j = 0, \ldots, k), \tag{4}$$

Where z_i, z_i' and $z_0^{(j)}$ are arbitrary real numbers.

Here we prove that, if $\{x_i\}_{i=0}^{n}$ and $\{y_i\}_{i=1}^{n}$ are the zeros of the Laguerre polynomial $L_n^{(k)}(x)$ and its derivative $L_n^{(k)'}(x)$, respectively, and $x_0 = 0$, then there exists a unique polynomial $P_m(x)$ of degree $2n+k-1$ that satisfies the above conditions. In Pál-type interpolation the function values are prescribed at the zeros of $w_n(x)$, while the derivative values are prescribed at the zeros of $w_n'(x)$. Thus the interpolational polynomial $P_m(x)$ is a modified Pál-type interpolational polynomial with $w_{n+k}(x) = x^k L_n^{(k)}(x)$.

2. Preliminaries

We have used some well known results of the Laguerre polynomial $L_n^{(k)}(x)$ which are as follows:
The differential equation of the Laguerre polynomial is given by

$$xD^2L_n^k(x) + (1+k-x)DL_n^k(x) + nL_n^k(x) = 0, \tag{5}$$

where *n* is a positive integer and $k > -1$.
For the roots of $L_n^{(k)}(x)$ we have

$$2\sqrt{x_j} = \frac{1}{\sqrt{n}}[j\pi + O(1)] \tag{6}$$

$$|L_n^{(k)'}(x_j)| \sim j^{-k-\frac{3}{2}} n^{k+1}, (0 < x_j \le \Omega, n = 1, 2, 3, \ldots) \tag{7}$$

$$|L_n^k(x)| = \begin{cases} x^{-\frac{k}{2}-\frac{1}{4}} O(n^{\frac{k}{2}-\frac{1}{4}}), & cn^{-1} \le x \le \Omega \\ O(n^k), & 0 \le x \le cn^{-1} \end{cases} \tag{8}$$

$$O(l_j(x)) = O(l_j^*(x)) = 1, \tag{9}$$

Now we also have some properties of fundamental polynomials of the Lagrange interpolation which are given in Szegö [9] as:

$$l_j(x)=\frac{L_n^{(k)}(x)}{L_n^{(k)\prime}(x_j)(x-x_j)}, \tag{10}$$

$$l_j^*(x)=\frac{L_n^{(k)'}(x)}{L_n^{(k)''}(y_j)(x-y_j)}. \tag{11}$$

$$l_j(x_i)=\delta_{ij}, \quad l_j^*(y_i)=\delta_{ij} \tag{12}$$

$$\left|\int_0^x l_j(t)dt\right|=\left|\int_0^x l_j^*(t)dt\right|=O(n^{-1}), \tag{13}$$

$$L_n^{(k)}(x)=L_n^{(k+1)}(x)-L_{n-1}^{(k+1)}(x) \tag{14}$$

Here the degree of the polynomial $l_j(x)$ is $n-1$ and the degree of the polynomial $l_j^*(x)$ is $n-2$.

3. Explicit Representation of Interpolatory Polynomial

Let the inter scaled nodal points be given by (1), where $\{x_i\}_{i=0}^n$ and $\{y_i\}_{i=1}^n$ are the zeros of the Laguerre polynomials $L_n^{(k)}(x)$ and $L_n^{(k)'}(x)$, respectively. Then, for the prescribed numbers $\{z_i\}_{i=1}^n$ and $\{z_i'\}_{i=1}^{n-1}$ there exists a unique polynomial $P_m(x)$ of degree $\leq 2n+k-1$ satisfying the conditions (2), (3), and (4).

The polynomial $P_m(x)$ is explicitly given by:

$$P_m(x)=\sum_{j=1}^{n} z_j U_j(x)+\sum_{j=1}^{n-1} z_j' V_j(x)+\sum_{j=0}^{k} z_0^{(j)} W_j(x) \tag{15}$$

Where $\{U_j(x)\}_{j=1}^n$, $\{V_j(x)\}_{j=1}^{n-1}$, and $\{W_j(x)\}_{j=0}^k$ are the polynomials having the degree $\leq 2n+k-1$. These polynomials are unique and satisfy the following conditions:

for $j=1,2,\ldots,n$

$$\begin{cases} U_j(x_i)=\delta_{ij}, & (i=1,2,\ldots,n) \\ U_j'(y_i)=0, & (i=1,2,\ldots,n-1) \\ U_j^l(0)=0, & (l=0,1,\ldots,k) \end{cases} \tag{16}$$

for $j=1,2,\ldots,n-1$

$$\begin{cases} V_j(x_i)=0, & (i=1,2,\ldots,n) \\ V_j'(y_i)=\delta_{ij}, & (i=1,2,\ldots,n-1) \\ V_j^l(0)=0, & (l=0,1,\ldots,k) \end{cases} \tag{17}$$

for $l=0,1,\ldots,k$

$$\begin{cases} W_k(x_i)=0, & (i=1,2,\ldots,n) \\ W_k'(y_i)=0, & (i=1,2,\ldots,n-1) \\ W_k^l(0)=\delta_{lk}, & (l=0,1,\ldots,k) \end{cases} \tag{18}$$

Here δ_{ij} is a Kronecker delta,

$$\delta_{ij}=\begin{cases} 1, & i=j \\ 0, & i\neq j. \end{cases} \tag{19}$$

The explicit forms of the $U_j(x), V_j(x)$, and $W_j(x)$ are given in the following lemma.

Lemma 1: For k and n positive integers, on the nodal points (1) the fundamental polynomial $\{U_j(x)\}_{j=1}^n$ of the interpolational problem in (1)-(4) satisfying the interpolatory condition (16) is given by:

For $j=1,2,\ldots,n$

$$U_j(x)=\frac{x^{k+1}l_j(x)L_n^{(k)'}(x)}{x_j^{k+1}L_n^{(k)'}(x_j)}-\frac{x^k L_n^{(k)}(x)}{x_j^{k+1}\left[L_n^{(k)'}(x_j)\right]^2}\int_0^x \frac{tL_n^{(k)''}(t)+(2+k-x_j)L_n^{(k)'}(t)}{t-x_j}dt \tag{20}$$

Where $l_j(x)$ is given by (10).

Proof: For $j = 1, 2, \ldots, n$, let

$$U_j^*(x) = u_1 x^{k+1} l_j(x) L_n^{(k)'}(x) + u_2 x^k L_n^{(k)}(x) \int_0^x \frac{t L_n^{(k)''}(t) + u_3 L_n^{(k)'}(t)}{t - x_j} dt \quad (21)$$

be a polynomial of degree $\leq 2n + k - 1$. We can easily check that $U_j^*(x)$ satisfies the equation (16) provided

$$u_1 = \frac{1}{x_j^{k+1} L_n^{(k)'}(x_j)} \quad (22)$$

and

$$u_2 = \frac{-u_1}{L_n^{(k)'}(x_j)} \quad (23)$$

Note that the $U_j^*(x)$ is a polynomial of minimal degree $2n + k - 1$, so the integrand in (20) must be a polynomial which implies $tL_n^{(k)''}(t) + u_3 L_n^{(k)'}(t) = 0$. Thus, by using the equations (14) and (11), we get $u_3 = 2 + k - x_j$. Hence,

$$U_j^*(x) \equiv U_j(x) \quad (24)$$

which completes the proof of the lemma.

Lemma 2: For k and n positive integers, on the nodal points (1) the fundamental polynomial $\{V_j(x)\}_{j=1}^{n-1}$ of the interpolational problem in (1)-(4) satisfying the interpolatory condition (17) is given by:
for $j = 1, 2, \ldots, n-1$,

$$V_j(x) = \frac{x^k L_n^{(k)}(x)}{(k+1) y_i^k L_n^{(k)}(y_i)} \int_0^x l_j^*(t) dt \quad (25)$$

Where $l_j^*(x)$ is given by (11).

Proof: For $j = 1, 2, \ldots, n-1$, let

$$V_j^*(x) = v_1 x^k L_n^{(k)}(x) \int_0^x l_j^*(t) dt \quad (26)$$

be a polynomial of degree $\leq 2n + k - 1$. We can easily check that $V_j^*(x)$ satisfies the equation (17) provided

$$v_1 = \frac{1}{(k+1) y_i^k L_n^{(k)}(y_i)} \quad (27)$$

Thus,

$$V_j^*(x) \equiv V_j(x) \quad (28)$$

which completes the proof of the lemma.

Lemma 3: For k and n positive integers, on the nodal points (1) the fundamental polynomial $\{W_j(x)\}_{j=0}^{k}$ of the interpolational problem in (1)-(4) satisfying the interpolatory condition (18) is given by:
for $j = 0, 1, \ldots, k-1$

$$W_j(x) = a_j(x) x^j L_n^{(k)}(x) L_n^{(k)'}(x) + x^{k-1} L_n^{(k)}(x) \times \\ \times \left[w_j - \int_0^x \frac{L_n^{(k)''}(t) a_j(t) + b_j(t) L_n^{(k)'}(t)}{t^{k-j}} dt \right] \quad (29)$$

$$W_k(x) = \frac{1}{k! L_n^{(k)}(0)} x^k L_n^{(k)}(x), \quad (30)$$

where $a_j(x)$ and $b_j(x)$ are the polynomials of degree at most $k - j - 1$.

Proof: For fixed $j \in \{0,1,\ldots,k-1\}$ we will find the polynomial $W_j(x)$ in the form

$$W_j(x) = a_j(x)x^j L_n^{(k)}(x)L_n^{(k)'}(x) + x^{k-1}L_n^{(k)}(x)c_n(x), \qquad (31)$$

where the degree of the polynomial $a_j^*(x)$ is $k-j-1$ and the degree of the polynomial $c_n(x)$ is n. Also it is clear that for $l=0,1,\ldots,j-1, W_j^{(l)}(0)=0$. We know that $L_n^{(k)}(x_i)=0$, so $W_j(x_i)=0$ for $i=1,2,\ldots,n$. The coefficients of the polynomial $a_j(x)$ are determined by the system

$$W_j^{(l)}(0) = \frac{d^l}{dx^l}\left[a_j(x)x^j L_n^{(k)}(x)L_n^{(k)'}(x)\right]_{x=0} = \delta_{jl} \qquad (32)$$

where $l = j,\ldots,k-1$.

Now for the constants w_j, use the equation $W_j^{(k)}(0)=0$ and get

$$w_j := c_n(0) = \frac{-1}{k!L_n^{(k)}(0)}\frac{d^k}{dx^k}\left[a_j(x)x^j L_n^{(k)}(x)L_n^{(k)'}(x)\right]_{x=0} \qquad (33)$$

Now use the conditions $L_n^{(k)'}(y_i)=0, W_j'(y_i)=0$, and

$$\frac{d}{dx}\left[x^k L_n^{(k)}(x)\right] = (n+k)x^{k-1}L_n^{(k)'}(x) \qquad (34)$$

we have

$$c_n'(y_i) = -(y_i)^{j-k} L_n^{(k)''}(y_i)a_j(y_i) \qquad (35)$$

this will imply the value of $c_n'(x)$ as:

$$c_n'(x) = -\frac{L_n^{(k)''}(x)a_j(x) + b_j(x)L_n^{(k)'}(x)}{x^{k-j}} \qquad (36)$$

where the polynomial $b_j(x)$ is of degree $k-j-1$. The function $c_n'(x)$ will be a polynomial if and only if

$$\frac{d^r}{dx^r}\left[L_n^{(k)''}(x)a_j(x) + b_j(x)L_n^{(k)'}(x)\right]_{x=0} = 0 \qquad (37)$$

for $r = 0,\ldots,k-j-1$.

By using these equations, we can uniquely determined the coefficients of $b_j(x)$. Now integrate the equation (35) to get $c_n(x) = c_n(0) + \int_0^x c_n'(t)dt$, use the value of $c_n(0)$ from (33) we get the desired result as the proof of the theorem.

Theorem 1: For some fixed integers k and $n \geq 1$ if $\{z_i\}_{i=1}^n, \{z_i'\}_{i=1}^{n-1}$, and $\{z_0^{(j)}\}_{j=0}^k$ are arbitrary real numbers, then on the nodal points (1) there exists a unique polynomial $P_m(x)$ having the at most degree $2n+k-1$ satisfying the equations (2), (3), and (4). The existing polynomial can be written as:

$$P_m(x) = \sum_{j=1}^{n} U_j(x)z_j + \sum_{j=1}^{n-1} V_j(x)z_j' + \sum_{j=0}^{k} W_j(x)z_0^{(j)}, \qquad (38)$$

where the fundamental polynomials $U_j(x), V_j(x)$ and $W_j(x)$ are defined in the previous Lemmas.

Proof: By Lemmas 1, 2, and 3, the polynomial $P_m(x)$, defined in the theorem's statement, holds the equations (2), (3), and (4), it implies that the existence of the polynomial is valid.

To prove the uniqueness let us consider the following problem: Find the polynomial $S_m(x)$ having the least possible degree $2n+k-1$ satisfying the following interpolatory conditions:

for $j = 1, 2, \ldots, n-1$

$$\begin{cases} S_m(x_i) = 0, & (i = 1,2,\ldots,n) \\ S_m'(y_i) = 0, & (i = 1,2,\ldots,n-1) \\ S_m^l(0) = 0, & (l = 0,1,\ldots,k) \end{cases}$$

After taking into consideration of these equations it can be seen that

$$S_m(x) = x^{k-1} L_n^{(k)}(x) b_n(x),$$

where $b_n(x)$ is a polynomial of degree at most n. Use the equation (34) and get

$$S_m'(y_i) = L_n^{(k)}(y_i) y_i^k b_n'(y_i) = 0 \quad (i = 1, \dots, n),$$

from which $b_n'(y_i) = 0$ implies $b_n'(x) \equiv 0$, thus $b_n(x) \equiv c$. Therefore,

$$S_m(x) = c x^k L_n^{(k)}(x),$$

but

$$\frac{d^k S_m}{dx^k}(0) = c k! L_n^{(k)}(0) = 0.$$

Since $L_n^{(k)}(0) \neq 0$, therefore $c = 0$, hence $S_m(x) \equiv 0$. This proves that the polynomial $P_m(x)$ is unique. Now we state our main theorem.

Theorem 2: Assuming that the interpolatory function $f: R \to R$ is continuous as well as differentiable such that $C(m) = \{ f(x) : f(x) = O(x^m) \text{ as } x \to \infty; \}$ where m is a non negative integer, f is continuous function in the interval $[0, \infty)$, then for each $f \in C(m)$ and a non negative k,

$$P_m(x) = \sum_{j=1}^{n} z_j U_j(x) + \sum_{j=1}^{n-1} z_j' V_j(x) + \sum_{j=0}^{k} z_0^{(j)} W_j(x) \tag{39}$$

satisfies the relation:

$$|P_m(x) - f(x)| = O(n^{-1}) \omega \left(f, \frac{\log n}{\sqrt{n}} \right), \qquad for \;\; 0 \le x \le c n^{-1} \tag{40}$$

$$|P_m(x) - f(x)| = O(n^{-1}) \omega \left(f, \frac{\log n}{\sqrt{n}} \right), \qquad for \;\; c n^{-1} \le x \le \Omega \tag{41}$$

here ω represents the modulus of continuity.

Before proving the theorem 2, first estimate the values of the following fundamental polynomials, which are listed below:

4. Estimation of the Fundamental Polynomials

First estimate the values of the following fundamental polynomials, which are listed below:

Theorem 3: Let us assume the fundamental polynomial $U_j(x)$, for $j = 1, 2, \dots, n$ is presented by:

$$U_j(x) = \frac{x^{k+1} l_j(x) L_n^{(k)\prime}(x)}{x_j^{k+1} L_n^{(k)\prime}(x_j)} - \frac{x^k L_n^{(k)}(x)}{x_j^{k+1} \left[L_n^{(k)\prime}(x_j) \right]^2} \int_0^x \frac{t L_n^{(k)\prime\prime}(t) + (2 + k - x_j) L_n^{(k)\prime}(t)}{t - x_j} dt \tag{42}$$

then we have

$$\sum_{j=1}^{n} |U_j(x)| = O(n^{-3}), \qquad \text{for } \; 0 \le x \le \Omega \tag{43}$$

Proof: From the polynomial $U_j(x)$ we have

$$\sum_{j=1}^{n} |U_j(x)| \le \sum_{j=1}^{n} \frac{|x^{k+1}| \, |l_j(x)| \left| L_n^{(k)\prime}(x) \right|}{|x_j^{k+1}| \left| L_n^{(k)\prime}(x_j) \right|} + \sum_{j=1}^{n} \frac{|x^k| \left| L_n^{(k)}(x) \right|}{|x_j^{k+1}| \left[L_n^{(k)\prime}(x_j) \right]^2} \left| \int_0^x \frac{t L_n^{(k)\prime\prime}(t) + (2 + k - x_j) L_n^{(k)\prime}(t)}{t - x_j} dt \right| \tag{44}$$

Let

$$I = \left| \int_0^x \frac{t L_n^{(k)\prime\prime}(t) + (2 + k - x_j) L_n^{(k)\prime}(t)}{t - x_j} dt \right|$$

To evaluate I, let

$$\frac{L_n^{(k)}(x)}{x-x_j} = d_{j,n-1}x^{n-1} + d_{j,n-2}x^{n-2} + d_{j,n-3}x^{n-3} + \cdots + d_{j,0} \tag{45}$$

$$L_n^{(k)}(x) = (x-x_j)d_{j,n-1}x^{n-1} + d_{j,n-2}x^{n-2} + d_{j,n-3}x^{n-3} + \cdots + d_{j,0} \tag{46}$$

To find the values of the coefficients, use the general form of the Laguerre polynomial

$$L_n^{(k)}(x) = \sum_{\mu=0}^{n} \binom{n+k}{n-\mu} \frac{(-x)^{\mu}}{\mu!} \tag{47}$$

and get,

$$a_{j,n-1} = \frac{(-1)^n}{n!} \text{ and } a_{j,n-2} = \frac{(-1)^n}{n!}[x_j - n(n+k)].$$

Now let,

$$\int_0^x \frac{tL_n^{(k)''}(t) + (2+k-x_j)L_n^{(k)'}(t)}{t-x_j}dt = \sum_{i=0}^{n} A_{j,i}L_i^{(k)}(x) \tag{48}$$

on comparing the coefficient with the equation (44) we have

$$A_{j,n} = 1 + \frac{x_j}{n}.$$

Thus, by substituting the value of coefficient and using the equations (6), (7), (8) and (9), we get the desired result.

Theorem 4: Let us assume the fundamental polynomial $V_j(x)$, for $j = 1, 2, \ldots, n-1$ is presented by:

$$V_j(x) = \frac{x^k L_n^{(k)}(x)}{(k+1)y_i^k L_n^{(k)}(y_i)} \int_0^x l_j^*(t)dt \tag{49}$$

then we have

$$\sum_{j=1}^{n-1} |V_j(x)| = O(n^{-1}), \text{ for } 0 \le x \le \Omega \tag{50}$$

Proof: From the polynomial $V_j(x)$ we have:

$$\sum_{j=1}^{n-1} |V_j(x)| \le \sum_{j=1}^{n-1} \frac{|x^k| \left|L_n^{(k)}(x)\right|}{|(k+1)| \left|y_i^k\right| \left|L_n^{(k)}(y_i)\right|} \left|\int_0^x l_j^*(t)dt\right| \tag{51}$$

by using the equations (6), (8), and (13) we get the desired result, $\sum_{j=1}^{n-1} |V_j(x)| = O(n^{-1})$, for $0 \le x \le \Omega$.

$f: R \to R$ is continuous as well as differentiable such that $C(m) = \{f(x): f(x) = O(x^m) \text{ as } x \to \infty;\}$ where m is a non negative integer, f is continuous function in the interval $[0, \infty)$, then for each $f \in C(m)$ and a non

Remark: Let $C(m) = \{f(x): f$ is continuous in $[0, \infty)$, $f(x) = O(x^m)$ as $x \to \infty;\}$ where $m \ge 0$ is an integer. Then, by Szegö [12] Theorem 14.7,

$$lim_{n \to \infty} |f(x) - H_n^{(\alpha)}(f,x)|_I = 0 \tag{52}$$

where $I \subset (0, \infty)$ for $\alpha \ge 0$, or $I \subset (0, \infty)$ for $-1 < \alpha < 0$. Also note that there is a function in $C(m)$ such that $\{H_n^{(\alpha)}(f,x)\}$ diverges for $\alpha \ge 0$ at $x = 0$. And for the convergence rate, we have:

$$\left|f(x) - H_n^{(\alpha)}(f,x)\right|_I = \begin{cases} O\left(\omega\left(f, n^{-1-\alpha}\right)\right); & -1 < \alpha < 0 \\ O\left(\omega\left(f, \frac{logn}{\sqrt{n}}\right)\right); & \alpha \ge -\frac{1}{2} \end{cases} \tag{53}$$

Proof of the main theorem 2:

Let us suppose that $A_n(x)$ be a polynomial of degree $\le 2n+k-1$ and $P_m(x)$ be given by (15). Note that $P_m(x)$ is exact for every fundamental polynomial of degree $\le 2n+k-1$; therefore,

$$A_n(x)=\sum_{j=1}^{n}A_n(x_j)U_j(x)+\sum_{j=1}^{n-1}A_n'(y_j)V_j(x)+\sum_{j=0}^{k}A_n(x_0)W_j(x) \tag{54}$$

from equations (15) and (54) we get

$$\left|f(x)-P_m(x)\right|\le\left|f(x)-A_n(x)\right|+\left|A_n(x)-P_m(x)\right| \tag{55}$$

$$\le |f(x)-A_n(x)|+\sum_{j=1}^{n}|f(x_j)-A_n(x_j)|\,|U_j(x)|$$
$$+\sum_{j=1}^{n-1}\left|f'(y_j)-A_n'(y_j)\right|\,|V_j(x)|$$
$$+\sum_{j=0}^{k}\left|f^l(x_0)-A_n^l(x_0)\right|\,|W_j(x)| \tag{56}$$

Thus, equation (55) and the conclusions of theorem 3, and 4 complete the proof of the theorem 2.

6. Conclusions

In this paper, we have proved the existence, uniqueness, explicit representation, and order of convergence of the given interpolatory problem when the roots $\{x_i\}_{i=1}^{n}$ and $\{y_i\}_{i=1}^{n-1}$, are prescribed on the Laguerre polynomials $L_n^{(k)}(x)$ and its derivative $L_n^{(k)'}(x)$ respectively, with an additional condition. If $f:R\to R$ be a continuously differentiable interpolatory function, then there exists a polynomial $P_m(x)$ having the degree $\le 2n+k-1$ holding the equations (2), (3) and (4).

References

[1] Lénárd, M. On weighted $(0, 2)$-type interpolation, *Electron. Trans. Numer. Anal.* Vol. 25, pp. 206-223, 2006.

[2] Lénárd, M. Pál-type interpolation and quadrature formulae on Laguerre abscissas, *Math. Pannon.* 15/2 (2004), 265-274.

[3] Ojha, D.; Srivastava, R. An evaluation of weighted polynomial interpolation with certain conditions on the roots of Hermite polynomial, *Bull. Transilv. Univ. Bra¸sov Ser. III*: Mathematics and Computer Science, Vol. 1(63), No. 1-2021, 209-220.

[4] Pál, L. G. A new modification of the Hermite Fejér interpolation, *Anal. Math.* 1(1975), 197-205.

[5] Sebestyén, Z. F. Solution of the Szili-type conjugate problem on the roots of Laguerre polynomial, *Anal. Math.* 25(1999), 301-314.

[6] Srivastava, R.; Mathur, K. K. An interpolation process on the roots of Hermite polynomials $(0; 0, 1)$-interpolation on infinite interval, *Bull. Inst. Math. Acad. Sin. (N.S.)* Vol. 26, no. 3 (1998).

[7] Srivastava, R.; Singh, Y. (2018). An Interpolation Process on the Roots of Ultra spherical Polynomials, *Applications and Applied Mathematics: An International Journal (AAM)*, Vol. 13, Iss. 2, Article 32.

[8] Srivastava, R.; Vishwakarma, G. A new kind of weighted $(0; 0, 2)$ interpolation on Laguerre polynomial, *International Journal of Science and Research*, Vol. 4 Issue 10, October 2015.

[9] Szegö, G. *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publ., 23, New York, 1939., 4th ed. 1975.

[10] Xie, T. F. On the Pál's problem, *Chinese Quart J. Math.* 7 (1992), 48-54.