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**Research Article**

# On The Existence of Fixed Point in Non-Archimedean Fuzzy Strong B-Metric Spaces

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**ABSTRACT**

In our research paper, we introduce the concept of Non-Archimedean Fuzzy Strong b-Metric Spaces (NAFsb-MS), which serves as a generalization and extension of Fuzzy Metric Spaces. Specifically, Non-Archimedean Fuzzy Strong b-Metric Spaces lie between Fuzzy Strong b-Metric Spaces(Fsb-MS) and Fuzzy b-Metric Spaces(Fb-MS). Our primary focus is establishing a theorem related to the existence and uniqueness of fixed points within Complete NAFsb-MS. The rigorous proof of this theorem is presented in the paper. Specifically, we prove that a mapping in this space must possess a fixed point under certain conditions. Additionally, the study employs contractive mappings in NAFsb-MS. Notably, NAFsb-MS impose weaker conditions than fuzzy metric spaces. Consequently, they offer a better mathematical model for addressing real life than Fsb-MS. Furthermore, our paper investigates the completeness of NAFsb-MS. It explores the convergence properties of a sequence and the Cauchy sequence within NAFsb-MS space.

**Keywords:** t-norm, FMS, Fb-MS, Fsb-MS, NAFsb-MS.

## Introduction:

For centuries, mathematicians have been intrigued by the notion of fixed points. A fixed point is a specific point within a mathematical space that remains unchanged when subjected to certain transformations. These points hold significant importance across various mathematical disciplines, including analysis, topology, and dynamical systems. Their study is closely tied to stability, equilibrium, and optimization challenges.
In 1965, mathematician Lotfi A. Zadeh[1] introduced a significant concept known as fuzzy sets. Unlike traditional sets, which categorize elements as either belonging or not, fuzzy sets allow for gradual membership. Researchers and academics are becoming interested in fuzzy set theory since it is more useful than conventional set theory. This adaptability has found practical applications in diverse fields, from artificial intelligence to decision-making. Its uses are broad in a variety of fields, including image processing, fractals, engineering, navigation, and other scientific areas.
In 1975, mathematicians Kramosil and Michalek[2] first proposed the idea of FMS. These spaces extend classical metric spaces by accommodating fuzzy distances instead of crisp ones., The distance between any two points or locations in an FMS is characterized by a fuzzy number, effectively capturing the inherent uncertainty present in real-world measurements. This reinterpretation of probabilistic metric spaces has significantly broadened research possibilities and practical applications. George and Veeramani[3] slightly modified the concept of fuzzy metric space defined by[2] .
In 1988, Grabiec[4],[24] presented the Banach contraction which extended the renowned fixed point theorems of Banach and Edelstein to the realm of FMS. This outcome showed that a successful adaptation of the notion of fixed points to the fuzzy context was possible. Many mathematicians expanded the concept of fuzzy metric spaces by using the weaker triangle inequality, which is known as fuzzy b-metric spaces. For instance, see [5], [6], [7], [8], [9], [10], [11], [12],[13]. Many researchers work and contribute to this field see references[13], [14], [15], [16],[17],[25],[26].
In 2019, Oner and Sostak[13] investigated and studied Fsb-MS, which serves as a generalization of FMS. His work bridges the gap between fuzziness and strong b-metrics. In 2022, Shazia Kanwal[20] further explored the concept of Fsb-MS and proved some theorems related to fixed points in these spaces. We can see from the

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literature that many researchers work hard in the area of fuzzy metric spaces and use the concept of Non-Archimedean property in these spaces, see references[21], [22], [23].
Building upon this basis[20], we introduce the notion of NAFsb-MS. These spaces present the concept of Fsb-MS with non-Archimedean properties. In our paper, we replace the triangular inequality with a weaker condition by using the non-Archimedean property. We aim to explore their properties, establish fixed-point theorems, and contribute to the growing body of knowledge in this field.
The main objective of our paper can be described as follows:

- To introduce the notion of NAFsb-MS.
- To define essential terminology related to NAFsb-MS.
- To study the completeness of Fsb-MS, shedding light on their convergence properties.
- To establish some theorems related to fixed points in NAFsb-MS.
- To rigorously define properties specific to NAFsb-MS.

## Definitions:

**"Triangular Norm(t-norm)**[1]**:**
A mapping $f: [0,1] \times [0,1] \rightarrow [0,1]$ is called continuous t-norm if it satisfies the following axioms:
i. $f(x, y) = f(x, y)$, $\forall x, y \in [0,1]$ {Symmetry}
ii. $f(x_1, y_1) \leq f(x_2, y_2)$, if $x_1 \leq x_2$, $y_1 \leq y_2$ {Monotonicity}
iii. $f(f(x, y), z) = f(x, f(y, z))$ {Associativity}
iv. $f(1, a) = a$, $\forall a \in [0,1]$ {Boundary Condition}"

**"Fuzzy Metric Spaces**[3]**:**
Triple (X, M, *) is called fuzzy metric spaces, where X is an arbitrary non-empty set, * is t-norm and M is fuzzy set defined on $X \times X \times [0,\infty)$ such that for all x, y, z ∈X the following axioms satisfied
i. $M(x, y, 0) = 0$
ii. $M(x, y, t) = 1$, iff $x = y$, $t \geq 0$
iii. $M(x, y, t) = M(y, x, t)$ { Symmetry }
iv. $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ {Triangular Inequality}
v. $M(x, y, .) : [0, \infty) \rightarrow [0,1]$ is left continuous."

**"Fuzzy b-metric spaces**[11]**:**
A quadruple (X, M, *, b) is called fuzzy Strong b-metric space, where X is an arbitrary non-empty set, * is t-norm and M is a fuzzy set defined on $X \times X \times [0,\infty)$ if there exists $b \geq 1$ such that for all x, y, z ∈X the following axioms satisfied
i. $M(x, y, 0) = 0$
ii. $M(x, y, t) = 1$, iff $x = y$, $t \geq 0$
iii. $M(x, y, t) = M(y, x, t)$, $t \geq 0$ { Symmetry }
iv. $M(x, z, b.(t + s)) \geq M(x, y, t) * M(y, z, s)$ {Triangular Inequality}
v. $M(x, y, .) : [0, \infty) \rightarrow [0,1]$ is left continuous."

**"Fuzzy Strong b- Metric Spaces**[20]**:**
A quadruple (X, M, *, b) is called fuzzy Strong b-metric space, where X is an arbitrary non-empty set, $b \geq 1$ be a real no, * is t-norm and M is a fuzzy set defined on $X \times X \times [0,\infty)$ if there exists $b \geq 1$ such that for all x, y, z ∈X the following axioms satisfied
vi. $M(x, y, 0) = 0$
vii. $M(x, y, t) = 1$, iff $x = y$, $t \geq 0$
viii. $M(x, y, t) = M(y, x, t)$, $t \geq 0$ { Symmetry }
ix. $M(x, z, t + b. s) \geq M(x, y, t) * M(y, z, s)$ {Triangular Inequality}
x. $M(x, y, .) : [0, \infty) \rightarrow [0,1]$ is left continuous."

**"Non- Archimedean Fuzzy Strong b- Metric Spaces:**
A quadruple (X, M, *, b) is called Non –Archimedean Fuzzy Strong b-metric spaces, where X is an arbitrary non-empty set, $b \geq 1$ be a real no, * is t-norm, and M is a fuzzy set defined on $X \times X \times [0,\infty)$ such that for all x, y, z ∈X the following axioms satisfied
i. $M(x, y, 0) = 0$
ii. $M(x, y, t) = 1$, iff $x = y$, $t \geq 0$
iii. $M(x, y, t) = M(y, x, t)$, $t \geq 0$ { Symmetry }
iv. $M(x, z, \max\{t, b. s\}) \geq M(x, y, t) * M(y, z, s)$, $t \geq 0$, $s \geq 0$ {Triangular Inequality}
v. $M(x, y, .) : [0, \infty) \rightarrow [0,1]$ is left continuous."

## Remark:

"Let (X, M, *, b) be a Non-Archimedean Fuzzy Strong b-Metric Space:

i. A sequence $\{x_n\}$ in X is said to be convergent and converges to a point p∈ $X$ if $\lim_{n\to\infty}$ M $(p_n, p, t) = 1, \forall t \geq 0$.

ii. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for any ε > 0 and t > 0, $\exists n_0 \in N$ such that $M(x_n, x_m, t) < \varepsilon$, $\forall n, m \geq n_0$.

iii. A Non-Archimedean Fuzzy Strong b- metric space is called complete if every Cauchy sequence in it is convergent."

**"Banach contraction**[4]**:**
Let (X, d) be a metric space. A mapping f: X ⟶ X is known as Banach contraction on f if there is a positive real number 0 < α < 1 such that ∀x, y ∈ X:
d(fx, fy) ≤ αd(x, y)."

**"Kannan Contraction**[20]**:**
Let (X, d) be a metric space and f: X⟶ X be a mapping if ∃ α ∈ (0, 1/2) such that, for all $x_1, x_2 \in X$, we have
$d(fx_1, fx_2) \leq \alpha\{d(x_1, fx_1) + d(x_2, fx_2)\}$"

## Main Result:

**Example**

**1.** Consider a strong b-metric space (X, D, b) and mapping A: X×X×[0,∞)⟶[0,1] defined by

$$A(p1, p2, u) = \begin{cases} e^{-\frac{D(p1,p2)}{u}}, & u > 0 \\ 0, & u = 0 \end{cases}$$

Then, (X, A,^,b) is Non-Archimedean fuzzy strong b-metric space, where ^ is the minimum t-norm.
Solution: We shall only verify the iv axiom of NAFsb-MS as the remaining axioms are trivial.
For this $p_1, p_2, p_3 \in X$ and u,h > 0.
We assume that $A(p_1, p_2, u) \leq A(p_2, p_3, h)$
Then $e^{-\frac{D(p1,p2)}{u}} \leq e^{-\frac{D(p2,p3)}{h}}$
This implies $-\frac{D(p1,p2)}{u} \leq -\frac{D(p2,p3)}{h}$
$\frac{D(p2,p3)}{k} \leq \frac{D(p1,p2)}{u}$
$u\, D(p2,p3) \leq k\, D(p1,p2)$
On the contrary,

$$A\{p1, p3, \max(u + b.h)\} = e^{-\frac{D(p1,p3)}{\max\{u+b.h\}}}$$

$$\leq e^{-\frac{D(p1,p2)+b.D(p2,p3)}{\max\{u+b.h\}}}$$

We will show that $e^{-\frac{D(p1,p2)+b.D(p2,p3)}{\max\{u+b.h\}}} \geq e^{-\frac{D(p1,p2)}{u}}$
Hence we will obtain that $A\{p1, p3, \max(u + b.h)\} \geq A(p2, p3, u)$

$$= A(p1, p2, u)^A(p2, p3, h)$$

We remark that $e^{-\frac{D(p1,p2)+b.D(p2,p3)}{\max\{u+b.h\}}} \geq e^{-\frac{D(p1,p2)}{u}}$
This implies $\frac{D(p1,p2)}{u} \geq \frac{D(p1,p2)+b.D(p2,p3)}{\max\{u+b.h\}}$

$$\max\{u + b.h\}\ D(p1, p2) \geq u(D(p1, p2) + b.D(p2, p3))$$

This implies $u\, D(p2, p3) \leq k\, D(p1, p2)$
Which is true.

**2.** Let a strong b-metric space (X, D, t) and $A_D: X * X * [0, \infty) \to [0,1]$ defined by

$$A_D(p1, p2, t) = \begin{cases} S/s + D(p1, p2) & if\ s > 0 \\ 0 & if\ s = 0 \end{cases}$$

Then (X, $A_D$, ⋄ , t) is a NAFsb-Ms, where ⋄ is minimum t-norm.

## Theorem 1

Let (X, A, *, b) be a complete Non-Archimedean Fuzzy Strong b, * is continuous t-norm, f is a mapping $f: X \to X$ defined by A(fx, fy, bt) ≥ A(x, y, t) for all $x, y \in X$ where 0< b< 1,
A(x, y, t) is strictly increasing in variable t and

$\lim_{t\to\infty} A(x,y,t) = 1 \quad \forall x,y \in X$ ______________(1)
then there exists a unique fixed point of f.

**Proof**: let $\{x_n\}$ be any sequence in X and $x_0 \in X$ be any arbitrary element, So that
$x_n = fx_{n-1}$
$= f^n x_0$ $(n \in N)$ ______________(2)
$A(x_n, x_{n+1}, bt) = A(f^n x_0, f^{n+1} x_0, bt)$
$\geq A(f^{n-1}x_0, f^n x_0, t)$ $\{\because A(x,y,t)$ is strictly increasing$\}$
$= A(x_{n-1}, x_n, t)$
$= A(f^{n-1}x_0, f^n x_0, t)$
$\geq A(f^{n-2}x_0, f^{n-1}x_0, \frac{t}{b})$ ________________(3)
$= A(x_{n-2}, x_{n-1}, \frac{t}{b})$
$\geq A(x_0, x_1, \frac{t}{b^{n-1}})$
So,
$A(x_n, x_{n+1}, bt) \geq A(x_0, x_1, \frac{t}{b^{n-1}})$
By using axiom (iv) of Non-Archimedean Fuzzy Strong b-Metric Spaces and let for every $n \in N$ and positive integer r and t ≥0

$A(x_n, x_{n+r}, t) \geq A(x_n, x_{n+1}, t) * A(x_{n+1}, x_{n+r}, \frac{t}{2s})$
$\geq A(x_n, x_{n+1}, t) * A(x_{n+1}, x_{n+2}, \frac{t}{2s}) * A(x_{n+2}, x_{n+r}, \frac{t}{4s})$
$\geq A(x_n, x_{n+1}, t) * A(x_{n+1}, x_{n+2}, \frac{t}{2s}) * A(x_{n+2}, x_{n+3}, \frac{t}{4s}) * A(x_{n+3}, x_{n+r}, \frac{t}{4s^2})$
$\geq A(x_n, x_{n+1}, t) * A(x_{n+1}, x_{n+2}, \frac{t}{2s}) * A(x_{n+2}, x_{n+3}, \frac{t}{4s}) * A(x_{n+3}, x_{n+4}, \frac{t}{4s^2})$
$* A(x_{n+4}, x_{n+r}, \frac{t}{8s^2})$
$A(x_n, x_{n+r}, t) \geq A(x_n, x_{n+1}, t) * A(x_{n+1}, x_{n+2}, \frac{t}{2s}) * A(x_{n+2}, x_{n+3}, \frac{t}{4s}) * A(x_{n+3}, x_{n+4}, \frac{t}{4s^2})$
$* A(x_{n+4}, x_{n+5}, \frac{t}{8s^2}) * A(x_{n+5}, x_{n+r}, \frac{t}{8s^3})$ _________________(4)

$A(x_n, x_{n+r}, t) \geq A(x_n, x_{n+1}, t) * A(x_{n+1}, x_{n+2}, \frac{t}{2s}) * A(x_{n+2}, x_{n+3}, \frac{t}{4s}) * A(x_{n+3}, x_{n+4}, \frac{t}{4s^2})$
$* A(x_{n+4}, x_{n+5}, \frac{t}{8s^2}) * A(x_{n+5}, x_{n+r}, \frac{t}{8s^3}) * --- * A(x_{n+r-1}, x_{n+r}, \frac{t}{2^{r-1}s^{r-1}})$.
Using (3) we have
$A(x_n, x_{n+r}, t) \geq A(x_0, x_1, \frac{t}{b^n}) * A(x_0, x_1, \frac{t}{2b^{n+1}}) * A(x_0, x_1, \frac{t}{2^2 b^{n+2}}) * --- *$
$A(x_0, x_1, \frac{t}{2^{r-1}s^{r-1}b^{n+r-1}})$. __________________(5)
As $n \to \infty$, $b^n \to 0 \Rightarrow \frac{t}{b^n} \to \infty$
So by using (1)
$A(x_n, x_{n+r}, t) \geq 1*1*1---*1$ (r times)
$A(x_n, x_{n+r}, t) \geq 1$

$\{x_n\}$ is a Cauchy sequence and X is complete, So there exists a point p in X such that
$$\lim_{n\to\infty} x_n = p$$
By using axiom (4) of Non- Archimedean Fuzzy Strong b- Metric Spaces
$A(y, fy, t) \geq A(y, x_{n+1}, t) * A(x_{n+1}, fy, \frac{t}{2s})$
$\geq A(y, x_{n+1}, t) * A(fx_n, fy, \frac{t}{2s})$
$A(y, fy, t) \geq A(y, x_{n+1}, t) * A(x_n, y, \frac{t}{2sk})$
When limit $n \to \infty$
$A(y, fy, t) \geq A(y, y, t) * A(y, y, \frac{t}{2sk})$
$\geq 1*1$
$A(b, fb, t) \geq 1$
So, fy = y
Y is a fixed point of f.
Uniqueness: Let y, y* be two fixed points of mapping f, then
fy = y and fy* = y*

Now, A(y, y*, t) = A(fy, fy*, t)

$\geq A(y, y^*, \frac{t}{k})$

This is a contradiction to the condition M(x,y, t) is strictly increasing in variable t.

So $y = y^*$

**Example:**

Consider (X, A, *, b) be NAFsb-MS with X = [0, 1] and $A: X \times X \times [0, \infty) \rightarrow [0,1]$ defined by

$$A(p1, p2, s) = \begin{cases} \frac{s}{s + |p1 - p2|} & if\ s > 0 \\ 0 & if\ s = 0 \end{cases}$$

Let $f: X \rightarrow X$ be defined by $fp_1 = \frac{p_1}{12}$ and $b = \frac{1}{6}$ & * is the minimum t-norm. Then there exists a unique fixed point of f.

We have $A(fp_1, fp_2, bs) = \frac{s/_6}{s/_6 + |fp_1 - fp_2|}$

$= \frac{s/_6}{s/_6 + |p_1 - p_2|/_{12}}$

$= \frac{s}{s + |p_1 - p_2|/_2}$

$\geq \frac{s}{s + |p_1 - p_2|}$

$\frac{s}{s + \{|p_1 - p_2|\}} = A(p1, p2, s)$

This implies $A(fp_1, fp_2, s) \geq A(p1, p2, s)$. Hence f contains a unique fixed point.

## Theorem 2

Suppose (X, A, *, b) is a complete non-Archimedean fuzzy strong b-metric space, * is continuous t-norm defined as * = min{$x_1, x_2$}, f is a mapping $f: X \rightarrow X$ defined by

A(fx, fy, bt) ≥ A(x, fx, t) ∗A(y, fy, t) ______________(1)

for all $x, y \in X$ where 0< b< 1 and t ≥ 0, A(x, y, t) is strictly increasing in variable t and

$\lim_{t\rightarrow\infty} A(x, y, t) = 1 \quad \forall x, y \in X$ ______________(2)

then there exists a unique fixed point of f.

**Proof:** Consider $x_0 \in X$, then, $fx_0 \in X$. Let $x_1 \in X$ such that $x_1 = fx_0$.

By induction, we find a sequence $x_n = fx_{n-1}$, in X.

Now, $A(x_n, x_{n+1}, bt) = A(fx_{n-1}, fx_n, bt)$

$\geq A(x_{n-1}, fx_{n-1}, t) * A(x_n, fx_n, t)$

$\geq A(x_{n-1}, x_n, t) * A(x_n, x_{n+1}, t)$

Since A(x, y, t) is strictly increasing in variable t and bt < t, so we are unable to write

$A(x_n, x_{n+1}, bt) \geq A(x_n, x_{n+1}, t)$ ________________(3)

Therefore, $A(x_n, x_{n+1}, bt) \geq A(x_{n-1}, x_n, t) = A(fx_{n-2}, fx_{n-1}, t)$

$\geq A(x_{n-2}, fx_{n-2}, \frac{t}{b}) * A(x_{n-1}, fx_{n-1}, \frac{t}{b})$

$\geq A(x_{n-2}, x_{n-1}, \frac{t}{b}) * A(x_{n-1}, x_n, \frac{t}{b})$

$\geq A(x_{n-2}, x_{n-1}, \frac{t}{b})$

$\geq A(x_0, x_1, \frac{t}{b^{n-1}})$ ________________(4)

$A(x_n, x_{n+1}, bt) \geq A(x_0, x_1, \frac{t}{b^{n-1}})$

By using axiom (iv) of Non-Archimedean Fuzzy Strong b-Metric Spaces and let a positive integer r and t ≥ 0

$A(x_n, x_{n+r}, t) \geq A(x_n, x_{n+1}, t) * A(x_{n+1}, x_{n+r}, \frac{t}{2s})$

$\geq A(x_n, x_{n+1}, t) * A(x_{n+1}, x_{n+2}, \frac{t}{2s}) * A(x_{n+2}, x_{n+r}, \frac{t}{4s})$

$A(x_n, x_{n+r}, t) \geq A(x_n, x_{n+1}, t) * A(x_{n+1}, x_{n+2}, \frac{t}{2s}) * A(x_{n+2}, x_{n+3}, \frac{t}{4s}) *$

$A(x_{n+3}, x_{n+4}, \frac{t}{4s^2}) * A(x_{n+4}, x_{n+5}, \frac{t}{8s^2}) *$

$A(x_{n+5}, x_{n+r}, \frac{t}{8s^3}) * - - - * A(x_{n+r-1}, x_{n+r}, \frac{t}{2^{r-1}s^{r-1}b^{n+r-1}})$.

By inequality (4)

$A(x_n, x_{n+r}, t) \geq A(x_0, x_1, \frac{t}{2b^n}) * A(x_0, x_1, \frac{t}{2^2 sb^{n+1}}) * - - -$

$- - * A(x_0, x_1, \frac{t}{2^r s^{r-1} b^{n+r-1}})$.

As $n \to \infty, b^n \to 0 \Rightarrow \frac{t}{b^n} \to \infty$

$A(x_n, x_{n+r}, t) \geq 1*1*1---*1$ (r times)

$A(x_n, x_{n+r}, t) \geq 1$

This implies $\{x_n\}$ is a Cauchy sequence in X and X is complete, So there exists a point p in X such that

$$\lim_{n\to\infty} x_n = p$$

By using contractive condition

$A(fx_n, fy, bt) \geq A(x_n, fx_n, t) * A(y, fy, t)$

$\geq A(x_n, x_{n+1}, t) * A(y, fy, t)$

When $n \to \infty$,

$A(y, fy, bt) \geq A(y, y, t) * A(y, fy, t)$

$\geq 1 * A(y, fy, t)$

$A(y, fy, bt) \geq A(y, fy, t)$ __________(5)

This is a contradiction to the condition A(x,y, t) is strictly increasing in variable t.

Hence, fy = y. So, y is fixed point of mapping f.

Uniqueness: Let y, y* be two fixed points of mapping f, then

fy = y and fy* = y*

Now, A(y, y*, t) = A(fy, fy*, t)

$\geq A(fy, fy^*, \frac{t}{b}) * A(y^*, fy^*, \frac{t}{b})$

$\geq 1 * 1$

$A(y, y^*, t) \geq 1$

y = y*

where t ≥ 0 and 0 < b< 1. Then, there will be a unique fixed point of f.

**Example:**

Consider (X, A, *, b) be NAFsb-MS with X = [0, 1] and $A: X \times X \times [0, \infty) \to [0,1]$ defined by

$$A(p1, p2, s) = \begin{cases} \frac{s}{s + |p1 - p2|} & if\ s > 0 \\ 0 & if\ s = 0 \end{cases}$$

Let $f: X \to X$ be defined by $fp_1 = \frac{p_1}{60}$ and $b = \frac{4}{6}$ & * is the minimum t-norm. Then there exists a unique fixed point of f.

Without loss of generality, we let $p_1 > p_2$ and $A(p_1, fp_1, s) * A(p_2, fp_2, s) = A(p_1, fp_1, s)$. Then we have to prove that $A(fp_1, fp_2, bs) \geq A(p_1, fp_1, s)$.

As $p_1, fp_1 \epsilon$ [0,1]

$$\left|\frac{p_1 - p_2}{60}\right| \leq \left|\frac{p_1 + p_2}{60}\right|$$
$$= \frac{1}{6}\left|\frac{6p_1}{60} + \frac{6p_2}{60}\right|$$
$$\leq \frac{1}{6}\left|\frac{59p_1}{60} + \frac{59p_2}{60}\right|$$
$$\leq \frac{1}{6}\left\{\left|\frac{59p_1}{60}\right| + \left|\frac{59p_2}{60}\right|\right\}$$

This implies

$$6\left|\frac{p_1 - p_2}{60}\right| \leq \left|\frac{59p_1}{60}\right| + \left|\frac{59p_1}{60}\right|$$

Or (4) $\frac{6}{4}\left|\frac{p_1 - p_2}{60}\right| \leq \left|\frac{59p_1}{60}\right| + \left|\frac{59p_1}{60}\right|$

Since we know that if $x, y, z \geq 0\ and\ 2x \leq y + z\ then\ x < \max\{y, z\}$

So $\frac{1}{6}\left|\frac{p_1 - p_2}{60}\right| \leq \left|\frac{59p_1}{60}\right|$

Since $t > 0$, $s + \frac{6}{4}\left|\frac{p_1 - p_2}{60}\right| \leq \left|\frac{59p_1}{60}\right|$

This implies $\frac{s}{s + \frac{6}{4}\left|\frac{p_1 - p_2}{60}\right|} \leq \frac{s}{s + \left|\frac{p_1 - p_2}{60}\right|}$

Or $A\left(fp_1, fp_2, \frac{4}{6}s\right) \geq A(p_1, fp_2, s) = A(p_1, fp_1, s) * A(p_2, fp_2, s)$

So by the above theorem f has a unique fixed point.

## Conclusion:

In this paper, we have introduced the novel concept of NAFsb-MS. These spaces blend the notion of Fsb-MS with the flexibility of non-Archimedean properties. We proved some theorems related to the existence and uniqueness of fixed in NAFsb-MS. We adopted a more flexible structure by utilizing the non-Archimedean property, departing from the traditional triangular inequality. This concept allowed us to explore new directions while maintaining mathematical rigor. Our findings have practical implications across various fields, including optimization, decision-making, and data analysis. Researchers and practitioners can get benefit from NAFsb-MS spaces to model real-world phenomena with greater accuracy while considering uncertainties. This approach will assist researchers in solving a wide range of equations and inequalities in the field of FMS and NAFsb-MS. This work inspires future investigation, research, and useful discussion, and advances mathematical knowledge.
It's important to remember that not all fuzzy b-metric spaces possess the additional properties of a Non-Archimedean fuzzy strong b-metric space. This is a subclass of fuzzy b-metric spaces with an additional condition related to the non-Archimedean property.

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