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Research Article

Fixed Point Theorems In Extended B-Metric Spaces Using Rational Type Contraction

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ABSTRACT

In this paper, we prove common fixed-point theorems in extended complete b-metric spaces using rational type contraction for two self-mappings. Our results extend and improve the results proved by Mlaiki et al. [1] for a single self-mapping in extended complete b-metric space. We extend their results for two self-mappings without assuming the continuity of any mapping.

AMS Mathematics Subject Classification. 47H10, 54H25.

Keywords: Fixed point, rational contractions, self-mappings, b-metric space, b-metric space.

1. Introduction.

Banach [2] in demonstrated a highly consequential theorem in the context of complete metric spaces, establishing the existence of a unique fixed point. Since then, the fixed-point theory is one of the most important tools in many branches of science, economics, computer science, engineering and the development of nonlinear analysis.

Bourbaki [3] and Bakhtin [4] initiated the idea of b-metric spaces. Czerwik [5] gave an axiom which was weaker than the triangular inequality and formally defined a b-metric space with a view of generalizing the Banach [2] contraction mapping theorem. He introduced a function that adjusts the triangle inequality by replacing the constant based on specific point interactions. Kir and Kiziltune [6], Boriceanu [7], Bota [8], Pacurar [9] extended used this idea and proved fixed point theorems and its applications in b-metric spaces.

Fagin et al. [10] used relaxation in triangular inequality and called this new distance measure as non-linear elastic matching (NEM). Similar type of relaxed triangle inequality was also used in many fields. Inspired by all these applications, Kamran et al. [11] introduced the concept of extended b-metric space and generalized many pre-existing results in literature. Alqahtani [12] presented the extension of rational inequalities, and W. Shatanawi in [13] discussed new types of contractions in extended b-metric spaces.

In this paper, we extend and improve the results of Mlaiki et al. [1] and prove common fixed-point theorems for two self-mappings in extended complete b-metric spaces using rational type contraction without assuming the continuity of any mapping.

2. Preliminaries.

Definition 2.1 [4] Let X be a non empty set and $s \geq 1$ be a given real number.
A function $d_b : X \times X \to [0,\infty)$ is called b-metric if it satisfies the following properties for each $x, y, z \in X$ –
(b_1) $d_b(x,y) = 0 \Leftrightarrow x = y$;
$(b2)$ $d_b(x,y) = d(y,x)$;
$(b3)$ $d_b(x,z) \leq s[d_b(x,y) + d_b(y,z)]$.
The pair (X, d_b) is called a b-metric space.

Example 2.1. Let $X = l_p(R)$ with $0 < p < 1$ where $l_p(R) = \{\{x_n\} \subset R : \sum_{n=1}^{\infty}|x_n|^p < \infty\}$. Define $d_b : X \times X \to R^+$ as-

$$d_b(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}}$$

where $x = \{x_n\}, y = \{y_n\}$. Then (X, d_b) is a b-metric space with coefficient $s = 2^{\frac{1}{p}}$.

Example 2.2. Let $X = L_p[0,1]$ be the space of all real functions $x(t), t \in [0,1]$ such that $\int_0^1 |x(t)|^p dt < \infty$ with $0 < p < 1$. Define $d_b: X \times X \to R^+$ as

$$d_b(x,y) = \left(\int_0^1 |x(t) - y(t)|^p dt\right)^{\frac{1}{p}}$$

Then (X, d_b) is a b-metric space with coefficient $s = 2^{\frac{1}{p}}$.

The above examples show that the class of b-metric spaces is larger than the class of metric spaces. When $s = 1$, the concept of b-metric space coincides with the concept of metric space.

Definition 2.2 [14] Let (X, d_b) be a b-metric space. A sequence $\{x_n\}$ in X is said to be:
(I) Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$;
(II) Convergent if and only if there exist $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$ and we write $\lim_{n\to\infty} x_n = x$;
(III) The b-metric space (X, d_b) is complete if every Cauchy sequence is convergent.

Definition 2.3 [11] Let X be a non-empty set and $\theta: X \times X \to [1, \infty)$. A function $d_\theta: X \times X \to [0, \infty)$ is called an extended b-metric if for all $x, y, z \in X$ it satisfies:
$(d_\theta 1)$ $d_\theta(x,y) = 0$ iff $x = y$.
$(d_\theta 2)$ $d_\theta(x,y) = d_\theta(y,x)$.
$(d_\theta 3)$ $d_\theta(x,z) \le \theta(x,z)[d_\theta(x,y) + d_\theta(y,z)]$.
The pair (X, d_θ) is called an extended b-metric space.

Remark 2.1. If $\theta(x,z) = s$ for $s \ge 1$, then we obtain the definition of a b-metric space.

Example 2.3 Let $X = \mathbb{Z}^+$. Define $\theta: X \times X \to \mathbb{R}^+$ and $d_\theta: X \times X \to \mathbb{R}^+$ by

$$\theta(x,y) = x + y + 1$$

And

$$d_\theta(x,y) = |x| + |y|$$

Then (X, d_θ) is an extended b-metric space.

Example 2.4 Let $X = C([a,b], R)$ be the space of all continuous real valued functions define on $[a,b]$. Then X is complete extended b-metric space for $d_\theta(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|^2$ with $\theta(x,y) = |x(t)| + |y(t)| + 2$ where $\theta: X \times X \to [1, \infty)$.

Definition 2.4 [11] Let (X, d_θ) be an extended b-metric space.
(i) A sequence $\{x_n\}$ in X is said to converge to $x \in X$, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_n, x) < \epsilon$ for all $n \ge N$. In this case we write $\lim_{n\to\infty} x_n = x$.
(ii) A sequence $\{x_n\}$ in X is said to be Cauchy, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_m, x_n) < \epsilon$ for all $m, n \ge N$.

Definition 2.5 [11] An extended b-metric space (X, d_θ) is complete if every Cauchy sequence in X is convergent.

Lemma 2.1 [11] Let (X, d_θ) be an extended b-metric space. If d_θ is continuous, then every convergent sequence has a unique limit.

3. Main Result.

Theorem 3.1. Let $P, Q: X \to X$ be self-mappings with (X, d_t) be an extended complete b-metric space and for all distinct $x, y \in X$-

$$d_t(Px, Qy) \le \xi_1 d_t(x,y) + \xi_2 \frac{d_t(x,Px)d_t(y,Px) + d_t(y,Qy)d_t(x,Qy)}{d_t(x,Qy) + d_t(y,Px)}$$

where $d_t(x,Qy) + d_t(y,Px) \ne 0, 0 < \xi_1 + \xi_2 < 1, \xi_1, \xi_2 \in [0,1)$. Then P and Q have a unique common fixed point in X.

Proof. Let $s_0 \in X$ be arbitrary and $\{s_n\}$ be a sequence in X such that

$$s_{n+1} = Ps_n, s_{n+2} = Qs_{n+1}.$$

Then

$$\begin{aligned} d_t(s_{n+1}, s_{n+2}) = d_t(Ps_n, Qs_{n+1}) &\le \xi_1 d_t(s_n, s_{n+1}) + \xi_2 \frac{d_t(s_n, Ps_n)d_t(s_{n+1}, Ps_n) + d_t(s_{n+1}, Qs_{n+1})d_t(s_n, Qs_{n+1})}{d_t(s_n, Qs_{n+1}) + d_t(s_{n+1}, Ps_n)} \\ &= \xi_1 d_t(s_n, s_{n+1}) + \xi_2 \frac{d_t(s_n, s_{n+1})d_t(s_{n+1}, s_{n+1}) + d_t(s_{n+1}, s_{n+2})d_t(s_n, s_{n+2})}{d_t(s_n, s_{n+2}) + d_t(s_{n+1}, s_{n+1})} \\ &= \xi_1 d_t(s_n, s_{n+1}) + \xi_2 d_t(s_{n+1}, s_{n+2}) \end{aligned}$$

which implies

$$d_t(s_{n+1}, s_{n+2}) \le \frac{\xi_1}{1-\xi_2} d_t(s_n, s_{n+1}) = \xi d_t(s_n, s_{n+1})$$

where $\xi = \frac{\xi_1}{1-\xi_2} \in [0,1)$.
Applying it recursively, we get

$$d_t(s_{n+1}, s_{n+2}) \le \xi^n d_t(s_0, s_1).$$

Since $\xi \in [0,1)$, we have

$$\lim_{n\to\infty} d_t(s_{n+1}, s_{n+2}) = 0$$

Or

$$\lim_{n\to\infty} d_t(s_n, s_{n+1}) = 0.$$

Now for $m \ge 1$, using the triangular inequality, we have

$$\begin{aligned} d_t(s_n, s_{n+m}) &\ge t(s_n, s_{n+m})[d_t(s_n, s_{n+1}) + d_t(s_{n+1}, s_{n+m})] = t(s_n, s_{n+m})d_t(s_n, s_{n+1}) + t(s_n, s_{n+m})d_t(s_{n+1}, s_{n+m}) \\ &\le t(s_n, s_{n+m})\xi^n d_t(s_0, s_1) + t(s_n, s_{n+m})t(s_{n+1}, s_{n+m})[d_t(s_{n+1}, s_{n+2}) + d_t(s_{n+2}, s_{n+m})] \\ &= t(s_n, s_{n+m})\xi^n d_t(s_0, s_1) + t(s_n, s_{n+m})t(s_{n+1}, s_{n+m})\xi^{n+1} d_t(s_0, s_1) \\ &+ \cdots . + d_t(s_n, s_{n+m}) \ldots . t(s_{n+m-1}, s_{n+m})\xi^{n+m-1} d_t(s_0, s_1) = \xi^n d_t(s_0, s_1) \sum_{n+m-1}^{i=1} \xi^i \prod_{p=1}^{i} t(s_{n+p}, s_{n+m}). \end{aligned}$$

Using the ratio test, it can be deduced that the series $\sum_{n+m-1}^{i=1} \xi^i \prod_{p=1}^{i} t(s_{n+p}, s_{n+m})$ is convergent to some $S_m \in (0, \infty)$, we have

$$d_t(s_n, s_{n+m}) \le \xi^n d_t(s_0, s_1) S_m.$$

As $n \to \infty$, we conclude that the sequence $\{s_n\}$ is a Cauchy sequence in the extended complete b-metric space (X, d_t). Therefore there exists $s \in X$ such that

$$\lim_{n\to\infty} s_n = s.$$

To show

$$Ps = s.$$

We have

$$\begin{aligned} d_t(Ps, s) &\le t(Ps, s)[d_t(Ps, Qs_{n+1}) + d_t(Qs_{n+1}, s)] \\ &\le t(Ps, s)d_t(Qs_{n+1}, s) \\ &+ t(Ps, s)\left[\xi_1 d_t(s, s_{n+1}) + \xi_2 \frac{d_t(s, Ps)d_t(s_{n+1}, Ps) + d_t(s_{n+1}, Qs_{n+1})d_t(s, Qs_{n+1})}{d_t(s, Qs_{n+1}) + d_t(s_{n+1}, Ps)}\right] \\ &= t(Ps, s)d_t(s_{n+2}, s) + t(Ps, s)\left[\xi_1 d_t(s, s_{n+1}) + \xi_2 \frac{d_t(s, Ps)d_t(s_{n+1}, Ps) + d_t(s_{n+1}, s_{n+2})d_t(s, s_{n+2})}{d_t(s, s_{n+2}) + d_t(s_{n+1}, Ps)}\right] \end{aligned}$$

As $n \to \infty$, we have

$$d_t(Ps, s) \le t(Ps, s)\xi_2 d_t(s, Ps).$$

Since $\xi_2 \in [0,1)$, we have $d_t(Ps, s) = 0$. Hence

$$Ps = s.$$

Similarly, we can show

$$Qs = s.$$

Therefore P and Q have a common fixed point in X i.e.

$$Ps = Qs = s.$$

To show uniqueness of the fixed point, let $z \ne s$ be another fixed point of P and Q i.e.

$$Pz = Qz = z; Ps = Qs = s.$$

Then

$$d_t(z, s) = d_t(Pz, Qs) \le \xi_1 d_t(z, s) + \xi_2 \frac{d_t(z, Pz)d_t(s, Pz) + d_t(s, Qs)d_t(z, Qs)}{d_t(z, Qs) + d_t(s, Pz)} = \xi_1 d_t(z, s).$$

Since $\xi_1 \in [0,1)$, we have $d_t(z, s) = 0$ i.e. $z = s$.
This completes the proof.

If we put $Q = P$, we get the Theorem 2.1 of Mlaiki et al. [1] without continuity of P.

Corollary 3.1. Let $P: X \to X$ be self-mapping with (X, d_t) be an extended complete b-metric space and for all distinct $x, y \in X$-

$$d_t(Px,Py) \le \xi_1 d_t(x,y) + \xi_2 \frac{d_t(x,Px)d_t(y,Px) + d_t(y,Py)d_t(x,Py)}{d_t(x,Py) + d_t(y,Px)}$$

where $d_t(x,Py) + d_t(y,Px) \neq 0, 0 < \xi_1 + \xi_2 < 1, \xi_1, \xi_2 \in [0,1)$. Then P has a unique fixed point in X.

Theorem 3.2. Let $P, Q: X \to X$ be self-mappings with (X, d_t) be an extended complete b-metric space and for all distinct $x, y \in X$-

$$d_t(Px,Qy) \le \xi_1 d_t(x,y) + \xi_2 \frac{d_t(x,Px)d_t(x,Py) + d_t(y,Qy)d_t(Px,y)}{d_t(x,Qy) + d_t(y,Px)} + \xi_3 \frac{d_t(x,Px)d_t(y,Px) + d_t(y,Qy)d_t(x,Qy)}{d_t(x,Qy) + d_t(y,Px)}$$

where $d_t(x,Qy) + d_t(y,Px) \neq 0, 0 < \xi_1 + \xi_2 + \xi_3 < 1, \xi_1, \xi_2, \xi_3 \in [0,1)$. Then P and Q have a unique common fixed point in X.

Proof. Let $s_0 \in X$ be arbitrary and $\{s_n\}$ be a sequence in X such that

$$s_{n+1} = Ps_n, s_{n+2} = Qs_{n+1}.$$

Then

$$\begin{aligned} d_t(s_{n+1}, s_{n+2}) &= d_t(Ps_n, Qs_{n+1}) \\ &\le \xi_1 d_t(s_n, s_{n+1}) + \xi_2 \frac{d_t(s_n, Ps_n)d_t(s_n, Ps_{n+1}) + d_t(s_{n+1}, Qs_{n+1})d_t(Ps_n, s_{n+1})}{d_t(s_n, Qs_{n+1}) + d_t(s_{n+1}, Ps_n)} \\ &+ \xi_3 \frac{d_t(s_n, Ps_n)d_t(s_{n+1}, Ps_n) + d_t(s_{n+1}, Qs_{n+1})d_t(s_n, Qs_{n+1})}{d_t(s_n, Qs_{n+1}) + d_t(s_{n+1}, Ps_n)} \\ &\le \xi_1 d_t(s_n, s_{n+1}) + \xi_2 \frac{d_t(s_n, s_{n+1})d_t(s_n, s_{n+2}) + d_t(s_{n+1}, s_{n+2})d_t(s_{n+1}, s_{n+1})}{d_t(s_n, s_{n+2}) + d_t(s_{n+1}, s_{n+1})} \\ &+ \xi_3 \frac{d_t(s_n, s_{n+1})d_t(s_{n+1}, s_{n+1}) + d_t(s_{n+1}, s_{n+2})d_t(s_n, s_{n+2})}{d_t(s_n, s_{n+2}) + d_t(s_{n+1}, s_{n+1})} \\ &= \xi_1 d_t(s_n, s_{n+1}) + \xi_2 d_t(s_n, s_{n+1}) + \xi_3 d_t(s_{n+1}, s_{n+2}) \end{aligned}$$

which implies

$$d_t(s_{n+1}, s_{n+2}) \le \frac{\xi_1 + \xi_2}{1 - \xi_3} d_t(s_n, s_{n+1}) = \xi d_t(s_n, s_{n+1})$$

where $\xi = \frac{\xi_1+\xi_2}{1-\xi_3} \in [0,1)$.
Applying it recursively, we get

$$d_t(s_{n+1}, s_{n+2}) \le \xi^n d_t(s_0, s_1).$$

Since $\xi \in [0,1)$, we have

$$\lim_{n\to\infty} d_t(s_{n+1}, s_{n+2}) = 0$$

Or

$$\lim_{n\to\infty} d_t(s_n, s_{n+1}) = 0.$$

Now for $m \ge 1$, using the triangular inequality, we have

$$\begin{aligned} d_t(s_n, s_{n+m}) &\ge t(s_n, s_{n+m})[d_t(s_n, s_{n+1}) + d_t(s_{n+1}, s_{n+m})] = t(s_n, s_{n+m})d_t(s_n, s_{n+1}) + t(s_n, s_{n+m})d_t(s_{n+1}, s_{n+m}) \\ &\le t(s_n, s_{n+m})\xi^n d_t(s_0, s_1) + t(s_n, s_{n+m})t(s_{n+1}, s_{n+m})[d_t(s_{n+1}, s_{n+2}) + d_t(s_{n+2}, s_{n+m})] \\ &= t(s_n, s_{n+m})\xi^n d_t(s_0, s_1) + t(s_n, s_{n+m})t(s_{n+1}, s_{n+m})\xi^{n+1} d_t(s_0, s_1) \\ &+ \cdots . + d_t(s_n, s_{n+m}) \ldots . t(s_{n+m-1}, s_{n+m})\xi^{n+m-1} d_t(s_0, s_1) = \xi^n d_t(s_0, s_1) \sum_{n+m-1}^{i=1} \xi^i \prod_{p=1}^{i} t(s_{n+p}, s_{n+m}). \end{aligned}$$

Using the ratio test, it can be deduced that the series $\sum_{n+m-1}^{i=1} \xi^i \prod_{p=1}^{i} t(s_{n+p}, s_{n+m})$ is convergent to some $S_m \in (0, \infty)$, we have

$$d_t(s_n, s_{n+m}) \le \xi^n d_t(s_0, s_1) S_m.$$

As $n \to \infty$, we conclude that the sequence $\{s_n\}$ is a Cauchy sequence in the extended complete b-metric space (X, d_t). Therefore there exists $s \in X$ such that

$$\lim_{n\to\infty} s_n = s.$$

To show

$$Ps = s.$$

We have

$$\begin{aligned}
d_t(Ps,s) &\le t(Ps,s)[d_t(Ps,Qs_{n+1}) + d_t(Qs_{n+1},s)] \\
&\le t(Ps,s)d_t(Qs_{n+1},s) \\
&+ t(Ps,s)\left[\xi_1 d_t(s,s_{n+1}) + \xi_2 \frac{d_t(s,Ps)d_t(s,Ps_{n+1}) + d_t(s_{n+1},Qs_{n+1})d_t(Ps,s_{n+1})}{d_t(s,Qs_{n+1}) + d_t(s_{n+1},Ps)}\right. \\
&\left. + \xi_3 \frac{d_t(s,Ps)d_t(s_{n+1},Ps) + d_t(s_{n+1},Qs_{n+1})d_t(s,Qs_{n+1})}{d_t(s,Qs_{n+1}) + d_t(s_{n+1},Ps)}\right] \\
&\le t(Ps,s)d_t(s_{n+2},s) \\
&+ t(Ps,s)\left[\xi_1 d_t(s,s_{n+1}) + \xi_2 \frac{d_t(s,Ps)d_t(s,s_{n+2}) + d_t(s_{n+1},s_{n+2})d_t(Ps,s_{n+1})}{d_t(s,s_{n+2}) + d_t(s_{n+1},Ps)}\right. \\
&\left. + \xi_3 \frac{d_t(s,Ps)d_t(s_{n+1},Ps) + d_t(s_{n+1},s_{n+2})d_t(s,s_{n+2})}{d_t(s,s_{n+2}) + d_t(s_{n+1},Ps)}\right]
\end{aligned}$$

As $n \to \infty$, we have

$$d_t(Ps,s) \le t(Ps,s)\xi_3 d_t(s,Ps).$$

Since $\xi_3 \in [0,1)$, we have $d_t(Ps,s) = 0$. Hence

$$Ps = s.$$

Similarly, we can show

$$Qs = s.$$

Therefore P and Q have a common fixed point in X i.e.

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To show uniqueness of the fixed point, let $z \neq s$ be another fixed point of P and Q i.e.

$$Pz = Qz = z; Ps = Qs = s.$$

Then

$$\begin{aligned}
d_t(z,s) &= d_t(Pz,Qs) \\
&\le \xi_1 d_t(z,s) + \xi_2 \frac{d_t(z,Pz)d_t(z,Ps) + d_t(s,Qs)d_t(Pz,s)}{d_t(z,Qs) + d_t(s,Pz)} + \xi_3 \frac{d_t(z,Pz)d_t(s,Pz) + d_t(s,Qs)d_t(z,Qs)}{d_t(z,Qs) + d_t(s,Pz)} \\
&= \xi_1 d_t(z,s).
\end{aligned}$$

Since $\xi_1 \in [0,1)$, we have $d_t(z,s) = 0$ i.e. $z = s$.
This completes the proof.

If we put $Q = P$, we get the Theorem 2.2 of Mlaiki et al. [1] without continuity of P.

Corollary 3.2. Let $P: X \to X$ be a self-mapping with (X, d_t) be an extended complete b-metric space and for all distinct $x, y \in X$-

$$d_t(Px,Py) \le \xi_1 d_t(x,y) + \xi_2 \frac{d_t(x,Px)d_t(x,Py) + d_t(y,Py)d_t(Px,y)}{d_t(x,Py) + d_t(y,Px)} + \xi_3 \frac{d_t(x,Px)d_t(y,Px) + d_t(y,Py)d_t(x,Py)}{d_t(x,Py) + d_t(y,Px)}$$

where $d_t(x,Py) + d_t(y,Px) \neq 0, 0 < \xi_1 + \xi_2 + \xi_3 < 1, \xi_1, \xi_2, \xi_3 \in [0,1)$. Then P has a unique fixed point in X.

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