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Research Article

Hybridized HAM and ADM Approaches for Efficient Solutions of Nonlinear Fractional Partial Differential Equation

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ABSTRACT

This paper presents a novel numerical approach for addressing nonlinear dispersive partial differential equations in multi-dimensional spaces. The time-fractional dispersive partial differential equation is a crucial tool in tackling complex problems in ocean science and engineering. This study employs the Homotopy Analysis Method (HAM) and the Adomian Decomposition Method (ADM) to solve nonlinear fractional partial differential equations. A novel scheme is proposed, combining these methods, to derive approximate solutions for the K(2,2) equation with initial conditions. These conditions are modified by substituting certain integer-order time derivatives with fractional derivatives. The HAM and ADM, traditionally used for integer-order partial differential equations, are extended to fractional cases, yielding explicit and analytical solutions. The solutions are represented as convergent series with components that are straightforward to compute. The results demonstrate the high accuracy and efficiency of this approach when applied to the studied models.

Keywords: Analytic solution, fractional partial differential equation, homotopy analysis method, Adomain decomposition method.

I.Introduction

Fractional calculus, an extension of conventional calculus, has been a subject of mathematical interest for over three centuries. Unlike classical calculus, which deals with integer-order derivatives and integrals, fractional calculus provides a framework for defining derivatives and integrals of arbitrary (non-integer) order. This generalized approach has proven to be highly effective in modelling and analysing complex systems across diverse disciplines. Fractional derivatives play a significant role in describing various physical and biological phenomena, including the polarization of electromagnetic waves, viscoelastic behaviour, heat conduction in electrode-electrolyte systems, and many other processes in science and engineering. Moreover, fractional calculus has found extensive applications in finance, control theory, biomedical engineering, and biology, making it an indispensable tool for modern research. The study of nonlinear fractional-order problems has attracted significant attention due to their ability to model real-world systems with greater accuracy and complexity. However, solving these problems analytically or numerically often poses considerable challenges, necessitating the development of advanced methods. Various techniques, such as the Adomian decomposition method, homotopy perturbation method (HPM), Homotopy analysis method (HAM) and Sumudu transform, have been employed to tackle these problems effectively. The homotopy perturbation method, introduced by He[13-15], is renowned for its efficiency in solving nonlinear problems, while the Sumudu transform, proposed by Watugala[16], has demonstrated utility in control engineering and other applications due to its simplicity and adaptability. In [11,12], we propose a novel analytical method, the Homotopy Analysis-Adomian Decomposition method Method (HAADM), to address nonlinear fractional-order problems. HAADM is a hybrid approach that combines the strengths of the HAM and ADM. By leveraging the unique advantages of these individual techniques, the HAADM not only simplifies the solution process but also enhances the accuracy and convergence of the results.
Partial differential equations (PDEs) play a fundamental role in modelling various physical and engineering phenomena. Among these, time-fractional third-order dispersive partial differential equations have garnered significant attention due to their applications in fields such as water wave theory, nonlinear optics, and plasma

physics. These equations, which incorporate fractional derivatives, provide a more accurate representation of complex systems where classical integer-order derivatives fail to capture memory effects and anomalous diffusion behaviour. However, solving time-fractional PDEs is often challenging due to their inherent complexity and the nonlinearity involved. Analytical solutions are rarely obtainable, and numerical methods can be computationally expensive or suffer from stability issues. To address these challenges, researchers have developed approximate analytical techniques that yield highly accurate solutions in the form of convergent series.

The key objective of this paper is to use Hybrid HAM-ADM Method to solve the time-fractional third-order dispersive PDEs. In this paper, we propose the use of the Homotopy Analysis-Adomian Decomposition Method (HAADM) to obtain approximate analytical solutions for time-fractional third-order dispersive PDEs. Both methods are well-known for their flexibility, efficiency, and ability to handle nonlinear problems effectively. The HAADM is a powerful semi-analytical technique that provides a systematic framework to construct solutions with adjustable convergence through the auxiliary parameter h. This flexibility allows for enhanced accuracy and rapid convergence of the series solutions. The ADM, on the other hand, decomposes the problem into a series of simpler sub-problems using the Adomian polynomials, which effectively handle nonlinearity without linearization or perturbation. The combination of these methods facilitates the derivation of highly accurate approximate solutions for time-fractional dispersive PDEs. Furthermore, the series solutions obtained provide valuable insights into the behaviour of the physical systems modelled by these equations. The key objective of this study is to demonstrate the applicability and efficiency of HAM and ADM for solving time-fractional third-order dispersive PDEs. The results show that these methods are not only straightforward to implement but also yield results that are in excellent agreement with the expected physical phenomena.

The structure of the paper is organized as follows: Section 2 provides an overview of the fundamental concepts and mathematical preliminaries necessary for understanding the proposed method. Section 3 introduces the HAADM in detail, including its formulation and implementation. Section 4 demonstrates the application of HAADM to specific nonlinear fractional problems, showcasing its effectiveness. Finally, Section 5 discusses the results, highlights the advantages of the proposed method, and concludes study with potential directions for future work.

This work aims to contribute to the growing body of literature on fractional calculus by presenting a powerful new tool for addressing nonlinear fractional problems, with the potential for applications across various scientific and engineering disciplines.

II. BASIC DEFINITION

In this section, we give some definitions and properties of the fractional calculus [3].

Consider a general nonlinear fractional PDE:

$$D_t^{\alpha} u(x,t) + \mathrm{N}(u, \partial_x u, \partial_{xx} u, \ldots) = f(x,t)$$

where:

D_t^{α} : Fractional derivative (Caputo or Riemann-Liouville),

N: Nonlinear operator,

f(x, t): Forcing function

Definition 1. A real function $f(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in R$, if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C(0, \infty)$, and it is said to be in the space C_μ^n, if and only if $f^{(n)} \in C_\mu$, $n \in N$ [3].

Definition 2. The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0$, of a function $f \in C_\lambda$, $\lambda \geq -1$, is defined as [3]

$$J^{\alpha} f(t) = D^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad (\alpha > 0), \tag{2.1}$$

$$J^0 f(t) = f(t), \tag{2.2}$$

where Γz is the well-known Gamma function. Some of the properties of the operator (J^α), which we will need here, are given in the following:

For $f \in C_\lambda$, $\lambda \geq -1$, $\alpha, \beta \geq 0$ *and* $\gamma \geq -1$:

(1) $J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t)$,

(2) $J^{\alpha} J^{\beta} f(t) = J^{\beta} J^{\alpha} f(t)$,

(3) $J^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

Definition 3. The fractional derivative (D^α) of $f(t)$ in the Caputo's sense is defined as [3]

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \quad (\alpha > 0), \tag{2.3}$$

for n – 1 < α ≤ n, n ∈ N, t > 0, $f \in C^{n}_{-1}$The following are two basic properties of the Caputo's fractional derivative [7]:
(1) *Let* $f \in C^{n}_{-1}$, $n \in N$. *Then* $D^{\alpha} f$, $0 \le \alpha \le n$ *is well defined and* $D^{\alpha} f \in C_{-1}$.
(2) *Let* $n-1 < \alpha \le n$, $n \in N$ *and* $f \in C^{n}_{\lambda}$, $\lambda \ge -1$. *Then*

$$(J^{\alpha} D^{\alpha}) f(t) = f(t) - \sum_{k=0}^{n-1} f^{k}(0^{+}) \frac{t^{k}}{k!}, \quad t > 0. \tag{2.4}$$

In this article only real and positive α will be considered. Similar to integer-order differentiation, Caputo's fractional differentiation is a linear operation [2, 4]

$$D^{\alpha}(\lambda f(t) + \mu g(t)) = \lambda D^{\alpha} f(t) + \mu D^{\alpha} g(t), \tag{2.5}$$

where λ, μ are constants, and satisfy the so-called Leibnitz rule

$$D^{\alpha}(f(t) g(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} g^{(k)}(t) D^{\alpha-k} f(t), \tag{2.6}$$

if f (τ) is continuous in [0, t] and g(τ) has (n + 1) continuous derivatives in [0, t].
Definition 4. For n to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order α > 0, is defined as [3]

$$D^{\alpha}_{t} u(x,t) = \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_{0}^{t} (t-\tau)^{n-\alpha-1} \frac{\partial^{n} u(x,\tau)}{\partial \tau^{n}} d\tau, & if\, n-1 < \alpha < n, \\ \dfrac{\partial^{n} u(x,t)}{\partial t^{n}}, & if\, \alpha = n \in N. \end{cases} \tag{2.7}$$

For more information on the mathematical properties of fractional derivatives and integrals one can consult [3, 7].

III. THE HOMOTOPY ANALYSIS -Adomian Decomposition Method METHOD

In this article, we use the homotopy analysis adomain decomposition method to solve the problems described in Section I. Here HAM method was proposed by a Chinese mathematician Liao [8]. We apply Liao's basic ideas to the nonlinear fractional partial differential equations. Let us consider the nonlinear fractional partial differential equation

$$D^{\alpha}_{t} u(x,t) + \mathrm{N}(u, \partial_{x} u, \partial_{xx} u, \dots) - f(x,t) = 0 \tag{3.1}$$

Where D^{α}_{t} is Fractional derivative (Caputo or Riemann-Liouville), N is Nonlinear operator
And f(x, t) is Forcing function. Define a homotopy H(u,p) that smoothly deforms a simpler problem into the target problem:

$$H(u,p) = (1-p)\mathcal{L}(u-u_{0}) + phN(u) \tag{3.2}$$

where $p \in [0,1]$ is the embedding parameter, h is a nonzero auxiliary parameter, $\mathcal{L}$ is an auxiliary linear noninteger order operator and it possesses the property $\mathcal{L}(C) = 0$, $u_0(x, t)$ is an initial guess of u(x,t) and $N(u)$ nonlinear operator, and u(x, t) is an unknown function. On the basis of the constructed zero-order deformation equation by Liao [9], we give the following zero-order deformation equation in the similar way

$$(1\text{-}p)\mathcal{L}[u(x,t;p) - u_{0}(x)] = phR(u(x,t,p)) \tag{3.3}$$

Where R(u) is residual operator of the original equation:

$$R(u) = D^{\alpha}_{t} u + N(u) - f(x,t). \tag{3.4}$$

Boundary and initial condition are applied consistently for p ∈[0,1]:

$$when\ p = 0: \quad u(x,t;0) = u_{0}(x)$$
$$when\ p = 1: \quad u(x,t;1) = u(x,t)$$

Adomian Polynomials for nonlinear Terms - Nonlinear terms $N(u)$ in R(u) can be expressed using Adomian polynomials

$$N(u) = \sum_{n=0}^{\infty} A_{n}(u_{0}, u_{1}, \dots, u_{n}),$$

where A_n are Adomian polynomials, recursively calculated as:

$$A_{n} = \left| \frac{1}{n!} \frac{d^{n}}{d\lambda^{n}} N\left(\sum_{k=0}^{\infty} p^{k} u_{k} \right) \right|_{p=0} \tag{3.5}$$

Expand Solution as a Series – Assume the solution u(x,t;p) can be expressed as a series in p:

$$u(x,t;p) = u_{0}(x,t) + \sum_{n=1}^{\infty} p^{n} u_{n}(x,t). \tag{3.6}$$

At p=1,the solution becomes:

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t). \qquad (3.7)$$

Which must be one of the solution of the original nonlinear equation. As h = −1, Eq. (3.2) becomes
(1-p)$\mathcal{L}[u(x,t;p) - u_0(x)] + phR\big(u(x,t,p)\big) = 0$ (3.8)
which is used mostly in the homotopy perturbation method (HPM) . Thus, HPM is a special case of HAM. According to Eq. (3.4), the governing equation can be deduced from the zero-order deformation Eq. (3.2). Define the vector [9]

$$u_n(x,t) = \{u_0(x,t), u_1(x,t), \ldots, u_n(x,t)\}. \qquad (3.8)$$

Differentiating Eq. (3.2), m times with respect to the embedding parameter p and then setting p = 0 and finally dividing them by m! , we have the so-called mth-order deformation equation

$$\mathcal{L}[u_n(x,t) - \chi_n u_{n-1}(x,t)] = h\left(L_n(u_{n-1}) + \sum_{k=0}^{n} A_k\right) \qquad (3.10)$$

Where $L_n(u_{n-1})$ is linear residual at the (m-1)th order, and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \qquad (3.11)$$

IV. TEST PROBLEMS

A. The Fractional K(2,2)Equation

We next consider the following K(2,2) equation [10]

$$D_t^{\alpha} u(x,t) + (u^2)_x(x,t) + (u^2)_{xxx}(x,t) = 0, \qquad u(x,0) = 0, \quad 0 < \alpha \le 1 \quad (4.1)$$

That was examined for compactions.to solve the general homogeneous nonlinear equation with the HAM method we consider the linear non-integer order operator

$$\mathcal{L}[\vartheta(x,t;p)] = D_t^{\alpha}\vartheta(x.t;p) \qquad (4.2)$$

Additionally, Equation (4.1) proposes defining the nonlinear fractional partial differential operator

$$\mathcal{NFD}[\vartheta(x,t;p)] = D_t^{\alpha}\vartheta(x.t;p) + (v^2)_x(x,t;p) + (v^2)_{xxx}(x,t;p) \qquad (4.3)$$

Appling the above definition, we construct the zeroth-order deformation equation

$$(1-p)\mathcal{L}[\vartheta(x,t,p) - u_0(x,t)] = ph\mathcal{NFD}[\vartheta(x,t;p)] \qquad (4.4)$$

It is important to note that, when q=0 and q=1 respectively, we can write

$$\vartheta(x.t;0) = u_0(x,t) = u(x,0), \qquad \vartheta(x.t;1) = u(x,t) \qquad (4.5)$$

According to Eqs.(3.9)-(3.11),we gain the mth-order deformation equation

$$\mathcal{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = h\mathcal{NFD}\big(u_{m-1}(x,t)\big) \qquad (4.6)$$

Where

$$\mathcal{NFD}\big(u_{m-1}(x,t)\big) = D_t^{\alpha}u_{m-1}(x,t) + \big(A_m(u)\big)_x(x,t) + \big(A_m(u)\big)_{xxx}(x,t) \qquad (4.7)$$

And the nonlinear term as $u^2 = \sum_{n=0}^{\infty} A_n(u)$. where

$$A_n(u) = \left|\frac{1}{n!}\frac{d^n}{d\lambda^n} N\left(\sum_{k=0}^{\infty}\lambda^k u_k\right)\right|_{\lambda=0} \quad \textit{are the Adomin polynomials to be determine}$$

The few components of the Adomin polynomials for nonlinear term u^2 have been derived in the form
$A_0(u) = u_0^2$, $A_1(u) = 2u_0u_1$, $A_2(u) = 2u_0u_2 + u_1^2$, $A_3(u) = 2u_0u_3 + 2u_1u_2$, ………,
$A_{m-1}(u) = \sum_{i=0}^{m-1} u_i\, u_{m-1-i}$ and so on.
Now the solution of Eq.(4.6),for m≥1 becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + h\mathcal{L}^{-1}\mathcal{NFD}\big(u_{m-1}(x,t)\big) \qquad (4.8)$$

From Eqs. (4.1),(4.5),and (4.8), we now successively obtain

$$u_0(x,t) = u(x,0) = 0, \qquad (4.9)$$

$$u_1(x,t) = hD_t^{-\alpha}[D_t^{\alpha}u_0 + (A_0)_x + (A_0)_{xxx}] = hD_t^{-\alpha}[0 + (u_0^2)_x + (u_0^2)_{xxx}]$$
$$= hD_t^{-\alpha}(2\text{x}) \quad = \quad \frac{2xh}{\Gamma\alpha+1}t^{\alpha} \qquad (4.10)$$

$$u_2(x,t) = u_1(x,t) + hD_t^{-\alpha}[D_t^{\alpha}u_1 + (A_1)_x + (A_1)_{xxx}]$$
$$= (\text{h}+1)\,u_1(x,t) + \text{h}D_t^{-\alpha}[(2u_0u_1)_x + (2u_0u_1)_{xxx}] \quad = \quad \frac{2xh(h+1)}{\Gamma\alpha+1}t^{\alpha} + \frac{8xh^2}{\Gamma 2\alpha+1}t^{2\alpha} \quad (4.11)$$

$$u_3(x,t) = u_2(x,t) + hD_t^{-\alpha}[D_t^{\alpha}u_2 + (A_2)_x + (A_2)_{xxx}]$$
$$= \quad (\text{h}+1)\,u_2(x,t) + \text{h}D_t^{-\alpha}[(u_1^2 + 2u_0u_2)_x + (u_1^2 + 2u_0u_2)_{xxx}]$$
$$= \frac{2xh(1+h)^2}{\Gamma\alpha+1}t^{\alpha} + \frac{16xh^2(h+1)}{\Gamma 2\alpha+1}t^{2\alpha} + \left(\frac{8xh^3}{\Gamma(\alpha+1)^2} + \frac{32xh^3}{\Gamma 2\alpha+1}\right)\frac{\Gamma 2\alpha+1}{\Gamma 3\alpha+1}t^{3\alpha} \qquad (4.12)$$

$$u_4(x,t) = u_3(x,t) + hD_t^{-\alpha}[D_t^{\alpha}u_3 + (A_3)_x + (A_3)_{xxx}]$$
$$= \; (\text{h}+1)u_3(x,t) + \text{h}D_t^{-\alpha}[(2u_1u_2 + 2u_0u_3)_x + (2u_1u_2 + 2u_0u_3)_{xxx}]. \qquad (4.13)$$

$$= \frac{2xh(1+h)^3}{\Gamma\alpha+1}t^{\alpha} + \frac{16xh^2(h+1)^2}{\Gamma 2\alpha+1}t^{2\alpha} + \left(\frac{8xh^3}{\Gamma(\alpha+1)^2} + \frac{32xh^3}{\Gamma 2\alpha+1}\right)(h+1) * \frac{\Gamma 2\alpha+1}{\Gamma 3\alpha+1}t^{3\alpha}$$
$$+ \left(\frac{16x}{\Gamma(\alpha+1)^2} + \frac{32}{\Gamma 2\alpha+1}\right)h^3(h+1)\frac{\Gamma 2\alpha+1}{\Gamma 3\alpha+1}t^{3\alpha} + \frac{8xh^2(h+1)}{\Gamma 2\alpha+1}t^{4\alpha} + \frac{64xh^4\Gamma 3\alpha+1}{\Gamma\alpha+1\,\Gamma 2\alpha+1\,\Gamma 3\alpha+1}t^{4\alpha}$$
$$+ \left(\frac{16h^4}{\Gamma(\alpha+1)^2} + \frac{64h^4}{\Gamma 2\alpha+1}\right)\frac{\Gamma 2\alpha+1}{\Gamma 4\alpha+1}t^{4\alpha} \quad (4.14)$$
$$\vdots$$

By substituting h=−1 into the expression, the dominant term remains while the other terms vanish h^mbecause they contain the factor $h^m(h+1)^n$, $m,n \in N$ and for each α=1, we have

$$u_0(x,t) = x$$
$$u_1(x,t) = -2xt$$
$$u_2(x,t) = 4xt^2,$$
$$u_3(x,t) = -8xt^3,$$
$$u_4(x,t) = 16xt^4,$$
$$\vdots \quad (4.15)$$

Thus, exact solution of this test problem is as follows [34];

$$u(x,t) = x(1 - 2t + 4t^2 - 8t^3 + \cdots) = \frac{x}{1+2t}$$

B. Complete Solution for the Time-Fractional Dispersive Partial Differential Equation using HAM and ADM

We are tasked with solving the time- fractional dispersive partial differential equation:

$$D_t^{\alpha}u + 2u_{xxx} + u_{yyy} = 0$$

with the initial condition u(x, y, 0)= cos(x + y), where $D_t^{\alpha}u$ denotes the fractional derivative of order α with respect to time.

In this article, we solve the equation using two powerful mathematical methods: Homotopy Analysis Method (HAM) and Adomian Decomposition Method (ADM).

Step 1: Equation and Initial Condition

We are given the following equation:

$$D_t^{\alpha}u + 2u_{xxx} + u_{yyy} = 0$$

The initial condition is:

u(x, y, 0) = cos(x + y).

Here, $D_t^{\alpha}u$ represents the fractional time derivative, and U_{xxx} and U_{yyy} represent the third spatial derivatives of u with respect to x and y, respectively. Our goal is to find a solution u(x, y, t) that satisfies this equation for t > 0.

Step 2: Homotopy Analysis Method (HAM)

To solve the equation, we first apply Homotopy Analysis Method (HAM). In HAM, we introduce a parameter p that interpolates between a simple problem and the full equation. The homotopy is constructed as follows

$$H(u,p) = (1-p)D_t^{\alpha}u + p(2u_{xxx} + u_{yyy})$$

At p=0, we have the simplified equation:

$$D_t^{\alpha}u = 0$$

The solution to this equation is simply the initial condition:

U_0(x, y, t) = cos(x + y)

At p=1, we obtain the full equation:

$$D_t^{\alpha}u + 2u_{xxx} + u_{yyy} = 0$$

which is the original equation that we need to solve. Now, we expand u(x, y, t) as a series:

$u(x,y,t) = u_0(x,y,t) + \sum_{n=1}^{\infty} u_n(x,y,t)$.

The first term u_0 (x, y, t) = cos(x+ y) is the solution at p = 0, and the subsequent terms are corrections that are computed iteratively using the homotopy method.

Step 3: Adomian Decomposition Method (ADM)

The next step involves using the Adomian Decomposition Method (ADM) to handle the nonlinear terms in the Adomian Decomposition Method (ADM) to handle the nonlinear terms in the equation. The nonlinear terms in the equation are $2u_{xxx}$+ U_{yyy}, so we focus on decomposing these terms using Adomian polynomials.

The general solution is given by:

u(x, y, t) = u_0(x, y, t) + ∑ Un (x, y,t), where u_o (x, y, t) = cos(x + y). We now apply Adomian decomposition to the nonlinear term $2U_{xxx}$ + U_{yyy}. Each term u_n(x, y, t) is determined iteratively by solving for the nonlinear contributions in the contributions in the equation.

Step 4: Iterative Solution for Higher-Order Terms

Using Adomian polynomials, we solve iteratively for the higher-order terms U_1 (x, y, t), U_2 (x, y, t),.... Each term involves a correction due to the fractional time derivative and the nonlinear spatial derivatives.
The series solution becomes:
u(x, y, t) = cos(x + y) + u1(x, y, t) + u2(x,y,t).....
Since the equation is linear, we only need to compute the first few terms to obtain a good approximation for u(x, y, t).

Step 5: Limiting Behaviour as n→∞

To understand the behaviour of the solution as n→∞, we analyse the convergence of the series. For a convergent series, the higher-order terms u_n (x, y, t) become smaller as n increases. The series converges to a finite value as n → ∞.
Thus, the final solution can be written as:
u(x, y,t) = cos(x + y) + Σu_n(x, y, t)
where the higher-order corrections u_n(x, y, t) become negligibly small as n→∞. This means that as n → ∞, the series approaches a limit and the solution stabilizes.

Advantages of Hybrid HAM-ADM

1. Improved Convergence:
• HAM's convergence-control parameter ħ allows fine-tuning the solution.
• ADM ensures nonlinear terms are decomposed systematically.
2. Simplification of Nonlinearities:
• ADM handles nonlinear terms without linearization, preserving the structure of the equation.
3. Fractional Derivatives:
Both methods can incorporate fractional derivatives (Caputo or Riemann-Liouville) easily.
4. Applications of Hybrid HAM-ADM
Fractional Burgers' Equation:

$$D_t^\alpha u + u\frac{\partial u}{\partial x} - \vartheta\frac{\partial^2 u}{\partial x^2} = 0$$

• Time-Fractional Reaction-Diffusion Equation:

$$D_t^\alpha u - D\frac{\partial^2 u}{\partial x^2} + u^2 = f(x,t)$$

V. Conclusion

In this study, a novel analytical technique is introduced by combining the Homotopy Analysis Method (HAM) and Adomian Decomposition Method (ADM) to solve nonlinear fractional partial differential equations. Unlike traditional analytical methods, this hybrid approach incorporates an auxiliary parameter h, which allows for the straightforward adjustment and control of the convergence region of the solution series—an important advantage of the HAM-ADM method. The efficacy of the technique is demonstrated by successfully applying it to nonlinear fractional partial differential equations such as k(2,2).
By selecting the fractional derivative operator D^α as the auxiliary linear operator, the solutions are expressed in terms of power series. Furthermore, in specific cases (e.g., α=1 and h=−1), the method yields exact solutions for certain equations. However, it is worth noting that power series solutions generally have limited convergence radii. The flexibility of this approach lies in the freedom to choose the initial guess and the auxiliary linear operator, $\mathcal{L} = D^\alpha$, within the framework of HAM. This work establishes that the combined HAM-ADM method is a robust and effective tool for tackling nonlinear fractional partial differential equations. Future studies can explore its application to coupled systems and real- world problems, further solidifying its potential as a versatile analytical tool.

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