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Research Article

Altering Between Adequate Criterions For Existence Of Solution Of Linear System Using Banach's Theorem

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ABSTRACT

This study explores the presence and singularity of solutions for a system of linear equations through the application of the Banach Fixed Point Theorem. The primary aim is to illustrate how modifying the definition of a metric influences the conditions required for the presence and singularity of solutions in such systems. By examining these changes, the work sheds light on the flexibility of metric-based approaches and their impact on solving linear equations, potentially opening pathways for broader applications in mathematical analysis and computational methods.

Keywords: Metric space, fixed points, Banach space, Banach fixed point theorem.

1. Introduction and Preliminaries

The concept of Banach space was introduced by Stefan Banach [1], who formulated a fixed-point theorem for contraction mappings in 1922.This theorem, known as the Banach Fixed Point Theorem (BFPT) or Banach Contraction Principle (BCP), was later proved by Renato Caccioppoli [2] in 1931.
Since there are extensive applications of BCP in establishing the presence and singularity of solutions to initial value problems, integral equations, and systems of linear equations (as discussed in [3] and [4]), this work focuses specifically on utilizing BCP to verify the presence and singularity of solutions for systems of linear equations. A.H. Siddiqi [5] demonstrated the applicability of BCP to a system of n linear equations with n unknowns, while E.T. Copson [6] provided the generalized application of BCP to the infinite system of linear equations.

Definition 1.1 Metric Space

A metric space is a pair (A,ξ) where A is a set and ξ is a metric on A [or distance function on A], is a function on $A\times A$ such that $\forall\ a,b,c\in A$
We have $\xi: A\times A\to\mathbb{R}^+\cup\{0\}$
1) $\xi(a,b)\geq 0$
2) $\xi(a,b)=0\Leftrightarrow a=b$
3) $\xi(a,b)=\xi(b,a)$
4) $\xi(a,b)\leq\xi(a,c)+\xi(c,b)$

Definition 1.2 Banach Space. '*The Banach space is a normed linear space that is complete with respect to its norm.*'
In a Banach space, every Cauchy sequence of vectors converges to a limit within the space, making it a complete linear space with a metric that measures either the length of a vector or the distance between two vectors.

Definition 1.3 Contraction.

Let $(\mathbb{A},\xi)$ be a *metric space*; a linear transformation $T:(\mathbb{A},\xi)\to(\mathbb{A},\xi)$ is said to be a contraction mapping if,

$$\exists\ \alpha\in\mathbb{R}, 0\leq\alpha<1;$$
$$\ni\ \xi(T_a,T_b)\leq\alpha\cdot\xi(a,b);\forall a,b\in\mathbb{A}\qquad(1)$$

Theorem 1.4 "Let $(\mathbb{A}, \xi)$ be a complete metric space in which the distance between two points p and q is denoted by metric $d(p,q)$. And let $T: \mathbb{A} \to \mathbb{A}$ be a contraction, that is, there exists some number $\alpha \in (0,1)$

$$\ni \xi(T_p, T_q) \leq \alpha \cdot \xi(p,q); \forall p,q \in \mathbb{A} \tag{2}$$

Where $T_p = T(p)$ *and* $T_q = T(q)$; then T has a unique fixed point, i.e., there exists a unique $r \in (A,d)$ such that $T_r = r$."

Theorem 1.5 [Linear Equations].
"If a system

$$x = Cx + \beta \tag{3}$$

of n Linear Equations in n unknowns $x_1, x_2, \dots, x_n$ (the components of x) satisfies

$$\sum_{k=1}^{n} |c_{jk}| < 1; j = 1,2, \dots, n \tag{4}$$

It has precisely one Solution x, the solution can be obtained as the limit of the iterative sequence $(x_1, x_2, \dots, x_n)$; where x_0 is arbitrary and

$$x_{m+1} = Cx_m + \beta; m = 0,1, \dots \tag{5}$$

Where, $C = (c_{jk})$ *and* β given."
On the above theorem Banach's theorem is applied by choosing a metric ξ on a nonempty set $\mathbb{A}$ of all ordered n tuples of real numbers, which defined as

$$\xi(x,z) = \max_j |x_j - z_j| \tag{6}$$

And a linear map $T: \mathbb{A} \to \mathbb{A}$ by

$$y = T_x = Cx + \beta \tag{7}$$

Where, $C = (c_{jk})$ is a fixed real $n \times n$ square matrix and $\beta = (\beta_j) \in \mathbb{A}$ *a fixed vector*[1].
Here, $(\mathbb{A}, \xi)$ is complete space and linear map T become contraction map according to equation (1) as

$$\alpha = \max_j \sum_{k=1}^{n} |c_{jk}|$$

Since, $0 < \alpha < 1$ we will have

$$\sum_{k=1}^{n} |c_{jk}| < 1; j = 1,2, \dots n \tag{8}$$

and this is sufficient for convergence.
It is called row sum criterion because it involves row sums obtained by summing the absolute values of the elements in a row of C.

2. Main results

In this work, we have changed the definition of taken metric ξ with two other definitions given by [6] and got other two criterions for convergence of solution of system of linear equations.

2.1(Column Sum Criterion) To the metric in (6) there corresponds the condition (8). If we use on $\mathbb{A}$ the metric ξ_1 defined by

$$\boldsymbol{\xi_1(x,z) = \sum_{j=1}^{n} |x_j - z_j|}$$

show that instead of (8) we obtain the condition

$$\boldsymbol{\sum_{j=1}^{n} |c_{jk}| < 1; k = 1,2, \dots n}$$

Proof. The metric in (6) is

$$d(x,z) = \max_j |x_j - z_j|$$

and the corresponding condition (8) is

$$\sum_{k=1}^{n} |c_{jk}| < 1; j = 1,2, \dots n$$

Let, X be set of all ordered n-tuples of real numbers and $x, y, z \in \mathbb{A}$

[1] **A fixed vector:** Here all vectors of $\mathbb{A}$ are column vectors for ease of matrix multiplication.

$$\ni x = (x_1, x_2, \dots, x_n), y = (y_{1,} y_2, \dots y_n), z = (z_1, z_2, \dots z_n)$$

Let d_1 be the metric on X defined by

$$d_1(x,z) = \sum_{j=1}^{n} |x_j - z_j|$$

It is clear that $(\mathbb{A}, \xi_1)$ is complete.
Define $T: \mathbb{A} \to \mathbb{A}$ by

$$y = T_x = Cx + \beta$$

Where, $C = (c_{jk})$ is a fixed real $n \times n$ square matrix and $b = (\beta_j) \in \mathbb{A}$ *a fixed vector*.
Writing in components,

$$y_j = \sum_{k=1}^{n} c_{jk} x_k + \beta_j \,;\, j = 1,2,3, \dots, n$$

Setting $w = (w_j) = T_z$

$$\ni d_1(y,w) = d_1(T_x, T_z) = \sum_{j=1}^{n} |y_j - w_j|$$

$$= \sum_{j=1}^{n} \left| \sum_{k=1}^{n} c_{jk}(x_k - z_k) \right|$$

$$\le \max_k \sum_{j=1}^{n} |c_{jk}| \left[\sum_{k=1}^{n} |x_k - z_k| \right]$$

$$\le \max_k \sum_{j=1}^{n} |c_{jk}| \cdot d_1(x,z); k = 1,2, \dots, n$$

$$\therefore d_1(T_x, T_z) \le \alpha d_1(x,z)$$

Where

$$\alpha = \max_k \sum_{j=1}^{n} |c_{jk}| \,; k = 1,2, \dots, n$$

Thus, T becomes a contraction if

$$0 < \alpha < 1$$

$$\Rightarrow \alpha = \max_k \sum_{j=1}^{n} |c_{jk}| < 1; k = 1,2, \dots, n$$

Hence, under the metric ξ_1 we obtain the condition

$$\sum_{j=1}^{n} |c_{jk}| < 1; k = 1,2, \dots, n \qquad (9)$$

2.2(Square Sum Criterion) To the metric in (6) there corresponds the condition (8). If we use on $\mathbb{A}$ the metric ξ_2 defined by

$$\xi_2(x,z) = \left[\sum_{j=1}^{n} (x_j - z_j)^2 \right]^{1/2}$$

show that instead of (8) we obtain the condition

$$\sum_{j=1}^{n} \sum_{k=1}^{n} {c_{jk}}^2 < 1$$

***Solution*.** The metric in (6) is

$$\xi(x,z) = \max_j |x_j - z_j|$$

and the corresponding condition (8) is

$$\sum_{k=1}^{n} |c_{jk}| < 1; j = 1,2, \dots . n$$

Let, $\mathbb{A}$ be set of all ordered n-tuples of real numbers and $x, y, z \in \mathbb{A}$

$$\ni x = (x_1, x_2, \dots, x_n), y = (y_{1,} y_2, \dots, y_n), z = (z_1, z_2, \dots, z_n)$$

Let ξ_2 be the metric on $\mathbb{A}$ defined by

$$\xi_2(x,z) = \left[\sum_{j=1}^{n}(x_j - z_j)^2\right]^{1/2}$$

It is clear that $(\mathbb{A}, \xi_2)$ is complete.
Define $T: \mathbb{A} \to \mathbb{A}$ by

$$y = Tx = Cx + \beta$$

Where, $C = (c_{jk})$ is a fixed real $n \times n$ square matrix and $\beta = (\beta_j) \in X$ *a fixed vector*.
Writing in components,

$$y_j = \sum_{k=1}^{n} c_{jk}x_k + \beta_j \,;\, j = 1,2,3,\ldots,n$$

Setting $w = (w_j) = T_z$

$$\ni \xi_2(y,w) = \left[\sum_{j=1}^{n}(y_j - w_j)^2\right]^{1/2}$$

$$\therefore [\xi_2(y,w)]^2 = \sum_{j=1}^{n}(y_j - w_j)(y_j - w_j)$$

$$\therefore [\xi_2(T_x,T_z)]^2 = \sum_{j=1}^{n}(y_j)(y_j - w_j) - \sum_{j=1}^{n}(w_j)(y_j - w_j)$$

$$= \left\{\sum_{j=1}^{n}\left[\sum_{k=1}^{n} c_{jk}x_k + \beta_j\right]\left[\sum_{k=1}^{n} c_{jk}(x_k - z_k)\right]\right\} - \left\{\sum_{j=1}^{n}\left[\sum_{k=1}^{n} c_{jk}z_k + \beta_j\right]\left[\sum_{k=1}^{n} c_{jk}(x_k - z_k)\right]\right\}$$

$$= \left[\sum_{k=1}^{n} c_{jk}(x_k - z_k)\right]\left[\sum_{j=1}^{n}\sum_{k=1}^{n} c_{jk}(x_k - z_k)\right]$$

$$\therefore [\xi_2(T_x,T_z)]^2 = \left(\sum_{j=1}^{n}\sum_{k=1}^{n} {c_{jk}}^2\right)\left[\sum_{k=1}^{n}(x_k - z_k)^2\right]$$

$$\therefore \xi_2(T_x,T_z) = \left[\left(\sum_{j=1}^{n}\sum_{k=1}^{n} {c_{jk}}^2\right)\left[\sum_{k=1}^{n}(x_k - z_k)^2\right]\right]^{1/2}$$

$$\leq \left(\sum_{j=1}^{n}\sum_{k=1}^{n} {c_{jk}}^2\right)\left[\sum_{k=1}^{n}(x_k - z_k)^2\right]^{1/2}$$

$$= \alpha \cdot \xi_2(x,z)$$

Where; $\alpha = \sum_{j=1}^{n}\sum_{k=1}^{n} {c_{jk}}^2$

$$\therefore \xi_2(T_x,T_z) \leq \alpha \cdot \xi_2(x,z)$$

Hence, T is a contraction mapping if $\alpha < 1$

$$\Rightarrow \left(\sum_{j=1}^{n}\sum_{k=1}^{n} {c_{jk}}^2\right) < 1 \qquad (10)$$

3. Conclusion

As observed, altering the definition of the chosen metric leads to different criteria for convergence based on the specific properties of the metric. In the first modification, we obtain the **column sum criterion**, which is derived by summing the absolute values of the elements in each column of the $n \times n$ matrix associated with the given system of linear equations. This approach emphasizes the contribution of column-wise absolute sums to determine convergence.

In the second case, we arrive at the **square sum criterion**, which involves calculating the sum of the squares of the elements within the same $n \times n$ matrix. This method focuses on the squared magnitudes of the matrix elements, providing an alternative perspective on convergence conditions. These variations demonstrate how different metrics can offer unique insights and tools for analysing the convergence of iterative processes in solving linear systems.

References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae,* p. 133–181, 1922.
[2] C. Miranda, "Renato Caccioppoli," *Annali di Matematica Pura ed Applicata volume,* vol. 47, 1959.
[3] G. F. Simmons, Introduction to Topology and Morden Analysis, New York: McGraw Hill, 1963.
[4] E. Kreyszing , Introductory Functional Analysis with Aplication, Canada: John Wiley & Sons, 1978.
[5] A. H. Siddiqi, Functional Analysis with Applications, NewDelhi: Tata McGraw-Hill Publishing Company, 1986.
[6] E. T. Copson, Metric Space, NewDelhi: Universal Bookstall, 1996.