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Research Article

Some New Properties Of Balancing Numbers

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ABSTRACT

In this paper, we discuss some new properties of Balancing Numbers. We discuss the generalisation of a property, some known properties can be derive as a special case of this property. We investigate some congruences of Balancing Numbers.

Keywords: Balancing Number.
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1 Introduction:

Balancing number is a positive integer B which satisfies the equation: $1+2+3+\cdots+(B-1)=(B+1)+(B+2)+\cdots+(B+k)$, where k is called balancer. For example, $B=6,35,204,1189$ are balancing numbers and $k=2,14,84,492$ respectively are balancers. A. Behera and G.K Panda [1] introduced the concept of balancing numbers in 1998.
The search for balancing numbers is well known integer sequence was first initiated by Liptai [2]. He proved that there is no balancing number in the Fibonacci sequence other than 1. He further showed that sequence balancing numbers in odd natural numbers are the sums of two consecutive balancing numbers.
Behera and Panda [1], while accepting 1 as a balancing number, have set $B_0=1, B_1=6$, and so on, using the symbol B_n for the n^{th} balancing number. Panda [4] has relabel the balancing numbers by setting $B_1=1, B_2=6$ and so on to standardize the notation at par with Fibonacci numbers.
Some results established by Behera and Panda [1] can be stated with this new convention as follows:
The second order linear recurrence:
$B_{n+1}=6B_n-B_{n-1}$ for $n>2\cdots(1)$
The relation:
$B_n=B_{r+1}B_{n-r}-B_rB_{n-r-1}$ for $1\leqslant r\leqslant n-2\cdots(2)$
The Binet form:
$B_n=\frac{\lambda_1^n-\lambda_2^n}{\lambda_1-\lambda_2}$ for $n=1,2,\cdots\cdots(3)$
where $\lambda_1=3+\sqrt{8}$ and $\lambda_2=3-\sqrt{8}$
The relation:

$$B_{n+1}B_{n-1}=(B_n+1)(B_n-1)\cdots(4)$$

Now using (1), we can set $B-0=B_2-6B_1=6-6(1)=0$

2 Properties of Balancing Numbers:

We know that if n,r and k with $r\leqslant k$ are real numbers, then $(n+k-r)(n+r)-n(n+k)=r(k-r)$. In the following theorem we prove an analogous property of balancing numbers. This theorem also generalizes some known properties of Balancing numbers.

Theorem 2.1 $B_{n+k-r}B_{n+r}-B_nB_{n+k}=B_rB_{k-r}$ *for* $r\leqslant k$.
Proof: Using the Binet form (3)

$$\begin{aligned}
&B_{n+k-r}B_{n+r}-B_nB_{n+k}\\
&=\left(\frac{\lambda_1^{n+k-r}-\lambda_2^{n+k-r}}{\lambda_1-\lambda_2}\right)\left(\frac{\lambda_1^{n+r}-\lambda_2^{n+r}}{\lambda_1-\lambda_2}\right)-\left(\frac{\lambda_1^{n}-\lambda_2^{n}}{\lambda_1-\lambda_2}\right)\left(\frac{\lambda_1^{n+k}-\lambda_2^{n+k}}{\lambda_1-\lambda_2}\right)\\
&=\left(\frac{\lambda_1^{2n+k}-\lambda_1^{n+k-r}\lambda_2^{n+r}-\lambda_1^{n+r}\lambda_2^{n+k-r}+\lambda_1^{2n+k}}{(\lambda_1-\lambda_2)^2}\right)-\left(\frac{\lambda_1^{2n+k}-\lambda_1^{n}\lambda_2^{n+k}-\lambda_1^{n+k}\lambda_2^{n}+\lambda_2^{2n+k}}{(\lambda_1-\lambda_2)^2}\right)\\
&=\frac{\lambda_1^{n}\lambda_2^{n+k}-\lambda_1^{n+k-r}\lambda_2^{n+r}-\lambda_1^{n+r}\lambda_2^{n+k-r}+\lambda_1^{n+k}\lambda_2^{n}}{(\lambda_1-\lambda_2)^2}
\end{aligned}$$

$= \frac{\lambda_2^k - \lambda_1^{k-r}\lambda_2^r - \lambda_1^r\lambda_2^{k-r} + \lambda_1^k}{(\lambda_1-\lambda_2)^2} \qquad (\because \lambda_1\lambda_2 = 1)$

$= \left(\frac{\lambda_1^r - \lambda_2^r}{\lambda_1-\lambda_2}\right)\left(\frac{\lambda_1^{k-r} - \lambda_2^{k-r}}{\lambda_1-\lambda_2}\right)$

$= B_r B_{k-r}$ ■

Behera and Panda [1] have proved that $B_n = B_{r+1}B_{n-r} - B_r B_{n-r-1}$ for $1 \leqslant r \leqslant n-1$ and Panda [4] has proved that $B_{m+n}B_{m-n} = B_m^2 - B_n^2$. The following theorem shows that these two results are special cases of the Theorem 2.1.

Theorem 2.2 (1)$B_n = B_{r+1}B_{n-r} - B_r B_{n-r-1}$ *for* $1 \leqslant r \leqslant n-1$.
(2)$B_{m+n}B_{m-n} = B_m^2 - B_n^2$ for $m \geqslant n$.
(3)$B_{m+n} = B_m B_{n+1} - B_{m-1}B_n$.

Proof: (1) Replace n by 1 and k by $n-1$ in the Theorem 2.1, we have

$B_{1+n-1-r}B_{1+r} - B_1 B_{1+n-1} = B_r B_{n-1-r}$

$\therefore B_{n-r}B_{r+1} - B_n = B_r B_{n-r-1}$

$\therefore B_n = B_{n-r}B_{r+1} - B_r B_{n-r-1}$

(2) Replace n by $m-n$, k by $2n$ and r by n in the Theorem 2.1, we have

$$B_{m-n+2n-n}B_{m-n+n} - B_{m-n}B_{m-n+2n} = B_n B_{2n-n}$$
$$\therefore B_m B_m - B_{m-n}B_{M+n} = B_n B_n$$
$$\therefore B_m^2 - B_n^2 = B_{m+n}B_{m-n}$$

(3) Replace n by 1, r by n and k by $m+n-1$ in the Theorem 2.1, we have

$$\therefore B_{1+m+n-1-n}B_{1+n} - B_1 B_{1+m+n-1} = B_n B_{m+n-1-n}$$
$$\therefore B_m B_{n+1} - B_{m+n} = B_n B_{m-1}$$
$$\therefore B_m B_{n+1} - B_{m-1}B_n = B_{m+n} \quad ■$$

Theorem 2.3 $B_n B_{n+2} B_{n+k} B_{n+k+2} + 9B_k^2$ *is always a perfect square for* $k \in \boldsymbol{N}$.

Proof: Replace k by $k+2$ and r by k in the Theorem 2.1, we have

$B_{n+k+2-k}B_{n+k} - B_n B_{n+k+2} = B_k B_{k+2-k}$

$\therefore B_{n+2}B_{n+k} - B_n B_{n+k+2} = B_k B_2$

$\therefore B_{n+2}B_{n+k} = B_n B_{n+k+2} + 6B_k$

$\therefore B_n B_{n+2} B_{n+k} B_{n+k+2} + 9B_k^2 = B_n B_{n+k+2}(B_n B_{n+k+2} + 6B_k) + 9b_k^2$

$= (B_n B_{n+k+2})^2 + 6B_k(B_n B_{n+k+2}) + 9B_k^2$

$= (B_n B_{n+k+2} + 3B_k)^2$ or $(B_{n+2}B_{n+k} - 3B_k)^2$ ■

Theorem 2.4 $7 \cdot 5^{n-2} - 1 < B_n < 35 \cdot 6^{n-3} + 1$ *for* $n \geqslant 3$.

Proof: Using recurrence relation (1), we have

$B_n = 6B_{n-1} - B_{n-2}$ for $n \geq 3$

$= 5B_{n-1} + B_{n-1} - B_{n-2}$

$> 5B_{n-1} + 4 \qquad (\because B_{n-1} - B_{n-2} > 4$ forall $n \geq 3)$

By continuing this process up to $n-2$ steps, we have

$B_n > 5^{n-2}B_2 + 4(5^{n-3} + 5^{n-4} + \cdots 5^2 + 5 + 1)$

$= 6 \cdot 5^{n-2} + 4\left(\frac{5^{n-2}-1}{4}\right)$

$= 7 \cdot 5^{n-2} - 1$

Now again

$B_n = 6B_{n-1} - B_{n-2}$ for $n \geq 3$

$\leqslant 6B_{n-1} - 1 \qquad (\because B_{n-2} \geq 1)$

By continuing this process up to $n-2$ steps, we have

$B_n \leqslant 6^{n-2}B_2 - (6^{n-3} + 6^{n-4} + \cdots 6^2 + 6 + 1)$

$= 6 \cdot 6^{n-2} - \left(\frac{6^{n-2}-1}{5}\right)$

$< 6^{n-1} - \left(\frac{5 \cdot 6^{n-3} - 5}{5}\right)$

$= 6^{n-1} - 6^{n-3} + 1$

$= 35 \cdot 6^{n-3} + 1$

Thus $7 \cdot 5^{n-2} - 1 < B_n < 35 \cdot 6^{n-3} + 1$ for all $n \geqslant 3$ ■

S. J. Gajjar [5] has proved that if B_n is the n^{th} balancing number, then $B_n = u_n v_n$, where $v_n + \sqrt{2}u_n = (1+\sqrt{2})^n$. Also he has proved that $B_n = \frac{u_{2n}}{2}$. Using these results, we prove the following theorem.

Theorem 2.5 $B_n = \frac{1}{2}\sum_{k=1}^{n} 2^{k-1}\binom{2n}{2k-1}$

Proof: We know that $v_n + \sqrt{2}u_n = (1+\sqrt{2})^n$.

$$\therefore v_{2n} + \sqrt{2}u_{2n} = (1+\sqrt{2})^{2n}$$

By comparing the coefficient of $\sqrt{2}$ of both the sides, we have

$$u_{2n} = \binom{2n}{1} + 2\binom{2n}{3} + 2^2\binom{2n}{5} + \cdots 2^{n-1}\binom{2n}{2n-1} = \sum_{k=1}^{n} 2^{k-1}\binom{2n}{2k-1}$$

$$\therefore B_n = \frac{u_{2n}}{2} = \frac{1}{2}\sum_{k=1}^{n} 2^{k-1}\binom{2n}{2k-1} \blacksquare$$

Theorem 2.6 *For* $0 < r < k \quad B_{kn+r} \equiv \begin{cases} B_r \pmod{B_k} & if\ n\ iseven \\ -B_{k-1}B_r \pmod{B_k} & if\ n\ isodd \end{cases}$

Proof:

$$\begin{aligned} B_{kn+r} &= B_{k(n-1)+r+k} \\ &= B_{k(n-1)+r}B_{k+1} - B_{k(n-1)+r-1}B_k \\ &\equiv B_{k(n-1)+r}B_{k+1} \pmod{B_k} \\ &\equiv B_{k(n-1)+r}(6B_k - B_{k-1}) \pmod{B_k} \\ &\equiv -B_{k(n-1)+r}B_{k-1} \pmod{B_k} \\ &\equiv (-1)^2 B_{k(n-2)+r}B_{k-1}^2 \pmod{B_k} \\ &\equiv (-1)^2 B_{k(n-2)+r}(B_{k-2}B_k + 1) \pmod{B_k} \\ &\equiv (-1)^2 B_{k(n-2)+r} \pmod{B_k} \end{aligned}$$

After $\left\lfloor \frac{n}{2} \right\rfloor$ steps we have,

$$B_{kn+r} \equiv (-1)^{2\left\lfloor \frac{n}{2} \right\rfloor} B_{kn+r-2\left\lfloor \frac{n}{2} \right\rfloor k}$$

$$\equiv \begin{cases} (-1)^n B_r \pmod{B_k} & \text{if } n \text{ iseven} \\ (-1)^{n-1} B_{k+r} \pmod{B_k} & \text{if } n \text{ isodd} \end{cases}$$

$$\equiv \begin{cases} B_r \pmod{B_k} & \text{if } n \text{ iseven} \\ B_k B_{r+1} - B_{k-1}B_r \pmod{B_k} & \text{if } n \text{ isodd} \end{cases}$$

$$\equiv \begin{cases} B_r \pmod{B_k} & \text{if } n \text{ iseven} \\ -B_{k-1}B_r \pmod{B_k} & \text{if } n \text{ isodd} \end{cases} \blacksquare$$

3 Concluding Remarks:

In this paper, we have investigated some new properties of Balancing Numbers. We have found bound of Balancing Numbers. Also we have found congruence relation of Balancing Numbers. Such more relations and properties can also be found and it is an open area of research.

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